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Symmetries of sub-Riemannian surfaces

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ABSTRACT

We obtain some results on symmetries of sub-Riemannian surfaces. In case of a contact sub-Riemannian surface we base on invariants found by Hughen [15]. Using these invariants, we find conditions under which a sub-Riemannian surface does not admit symmetries. If a surface admits symmetries, we show how invariants help to find them. It is worth noting, that the obtained conditions can be explicitly checked for a given contact sub-Riemannian surface. Also, we consider sub-Riemannian surfaces which are not contact and find their invariants along the surface where the distribution fails to be contact.

0. Introduction

A sub-Riemannian manifold is a k-dimensional distribution endowed with a metric tensor on an n-dimensional manifold. At present sub-Riemannian geometry is intensively studied; this is motivated by applications in various fields of science (see, e.g. the book [1], where many applications of sub-Riemannian geometry are presented; also, for interesting examples, we refer the reader to [2–10], where applications to mechanics, thermodynamics, and biology are given). At the same time, various aspects of the theory of symmetries of sub-Riemannian manifolds are widely investigated because symmetries are always of great importance for applications [11,12]. Many papers are devoted to the theory of homogeneous (in part, symmetric) sub-Riemannian manifolds (see e.g. [13–16]). The main investigation tool in these papers is the Lie algebra theory as is usual when we study homogeneous spaces.

In the present paper we study symmetries of sub-Riemannian surfaces, i.e. of sub-Riemannian manifolds with $k = 2$ and $n = 3$. Our main goal is to give a practical tool (or an algorithmic procedure) for investigation of symmetries of a sub-Riemannian surface. The paper is organized as follows. In the first section we give in detail the construction of invariants of a contact sub-Riemannian surface using the Cartan reduction procedure (here we follow [15]) and show how to calculate them. In the second section we demonstrate how to apply invariants to finding symmetries of a contact sub-Riemannian surface. Finally, in the third section we consider a sub-Riemannian surface without assumption that it is contact and find invariants along the "singular surface", where the distribution fails to be contact.

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1. Contact sub-Riemannian surfaces

Let M be an n-dimensional manifold and $\varDelta$ be a k-dimensional distribution on M endowed with a metric tensor field

$$\forall p \in M, \quad \langle\cdot,\cdot\rangle_p : \varDelta_p \times \varDelta_p \to \mathbb{R}. \tag{1}$$

Then $(M, \varDelta, \langle\cdot,\cdot\rangle)$ is called a *sub-Riemannian manifold* [1].

In the present paper we consider a *sub-Riemannian surface* $\mathcal{S} = (M, \varDelta, \langle\cdot,\cdot\rangle)$, i.e. a two-dimensional distribution $\varDelta$ on a three-dimensional manifold M, where $\varDelta$ is endowed with a metric tensor field $\langle\cdot,\cdot\rangle$. In addition, we assume that *the distribution $\varDelta$ and the manifold M are oriented*. Note that we do not suppose that any metric on M is given.

Throughout the paper we will denote the Lie algebra of vector fields on a manifold N by $\mathfrak{X}(N)$, and the space of covector fields by $\mathfrak{X}(N)^*$. Also the space of r-forms on N will be denoted by $\varLambda^r(N)$.

1.1. *G-structure associated with a sub-Riemannian surface*

1.1.1. *Elements of theory of G-structures*

We recall here notions and results of the theory of G-structures we use in the present paper (for the details we refer the reader to [1,17]).

Tautological forms, pseudoconnection form, and structure equations. Let M be a smooth n-dimensional manifold, and $\pi : B(M) \to M$ be the coframe bundle of M.

On $B(M)$ the *tautological forms* $\theta^a \in \Omega^1(B(M))$ are defined as follows [17]. For a point $\xi \in B(M)$ ($\xi = \{\xi^a\}_{a=\overline{1,n}}$ is a coframe of T_pM, where $p = \pi(\xi)$), we set

$$\theta^a_\xi : T_\xi(B(M)) \to \mathbb{R}, \quad \theta^a_\xi(X) = \xi^a(d\pi(X)). \tag{2}$$

Now, on a neighborhood U of a point $p \in M$, take a coframe field $\eta = \{\eta^a\}$. This gives a trivialization $\alpha : \pi^{-1}(U) \to U \times GL(n)$: to a coframe ξ at $p \in U$ we assign $(p, g) \in U \times GL(n)$ such that $\xi^a = \tilde{g}^a_b \eta^b_p$, where $\|\tilde{g}^a_b\| = g^{-1}$.

For a coframe field η on U let us consider the pullback 1-forms $\bar{\eta}^a = d\pi^*\eta^a$ on $U \times GL(n) \cong \pi^{-1}(U) \subset B(M)$. Then

$$\theta^a_{(p,g)} = \tilde{g}^a_b \bar{\eta}^b_{(p,g)} = \tilde{g}^a_b d\pi^* \eta^b_p. \tag{3}$$

A *G-structure* $P \to M$ is a principal subbundle of $\pi : B(M) \to M$ with structure group $G \subset GL(n)$. The tautological forms on P are the restrictions of θ^a to P and will be denoted by the same letters.

Let us denote by $\mathfrak{g}$ the Lie algebra of the Lie group G. A *pseudoconnection form* ω on a G-structure $\pi : P \to M$ is a $\mathfrak{g}$-valued 1-form on P such that $\omega(\sigma(a)) = a$, where $\sigma(a)$ is the fundamental vector field ([17], Ch. I, Sec. 5) on P corresponding to $a \in \mathfrak{a}$.

Given a pseudoconnection form ω, we have *structure equations* on P:

$$d\theta^a = \omega^a_b \wedge \theta^b + T^a_{bc}\theta^b \wedge \theta^c \tag{4}$$

where the functions $T^a_{bc} : P \to \mathbb{R}$ uniquely determined by the Eq. (4) are called *torsion functions*, and the map $T : P \to \varLambda^2\mathbb{R}^n \otimes \mathbb{R}^n$, $\xi \to \{T^a_{bc}(\xi)\}$, is called the *torsion* of the pseudoconnection ω^a_b.

Structure function. Let us find how the torsion changes under change of the pseudoconnection. If ω^a_b, $\hat{\omega}^a_b$ are pseudoconnections on P, then $\mu^a_b = \hat{\omega}^a_b - \omega^a_b$ is a $\mathfrak{g}$-valued form on P with the property that $\mu(\sigma(a)) = 0$ for any $a \in \mathfrak{g}$. Then $\mu^a_b = \mu^a_{bc}\theta^c$.

$$\begin{aligned} d\theta^a &= \hat{\omega}^a_b \wedge \theta^b + \hat{T}^a_{bc}\theta^b \wedge \theta^c = (\omega^a_b + \mu^a_{bc}\theta^c) \wedge \theta^b + \hat{T}^a_{bc}\theta^b \wedge \theta^c = \omega^a_b \wedge \theta^b + (\hat{T}^a_{bc} - \mu^a_{[bc]})\theta^b \wedge \theta^c \\ &= \omega^a_b \wedge \theta^b + T^a_{bc}\theta^b \wedge \theta^c. \end{aligned} \tag{5}$$

It hence follows that

$$\hat{\omega}^a_b = \omega^a_b + \mu^a_{bc}\theta^c \Rightarrow \hat{T}^a_{bc} = T^a_{bc} + \mu^a_{[bc]}. \tag{6}$$

Let us define the Spencer operator δ from the space of tensors $T^2_1(\mathbb{R}^n)$ of type (2, 1) to the space $\varLambda^2(\mathbb{R}^n) \otimes \mathbb{R}^n$ as follows:

$$\delta : t^a_{bc} \in T^2_1(\mathbb{R}^n) \mapsto t^a_{[bc]} = \frac{1}{2}(t^a_{bc} - t^a_{cb}). \tag{7}$$

Note that $\mathfrak{g} \otimes (\mathbb{R}^n)^* \subset \mathfrak{gl}(n) \otimes \mathbb{R}^* \cong T^2_1(\mathbb{R}^n)$ and we will denote the restriction of δ to $\mathfrak{g} \otimes (\mathbb{R}^n)^*$ by the same letter δ. Thus, (6) can be rewritten as follows:

$$\hat{\omega}^a_b = \omega^a_b + \mu^a_{bc}\theta^c \Rightarrow \hat{T}^a_{bc} = T^a_{bc} + \delta(\mu^a_{bc}). \tag{8}$$

From (8) we conclude that *if $\delta : \mathfrak{g} \otimes (\mathbb{R}^n)^* \to \varLambda^2(\mathbb{R}^n) \otimes \mathbb{R}^n$ is a monomorphism, then, pseudoconnections ω^a_b, $\hat{\omega}^a_b$ with the same torsion T^a_{bc} coincide.*