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The theorem of Schur in the Minkowski plane

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ABSTRACT

We prove a Lorentzian analogue of the theorem of Schur for spacelike (or timelike) curves in the Minkowski plane.

1. Introduction

A classical theorem in differential geometry of curves in Euclidean space $\mathbb{E}^3$ compares the lengths of the chords of two curves, one of them being a planar convex curve [1–3].

Theorem 1 (*Schur*)**.** *Let $\alpha_1, \alpha_2 : [0, L] \to \mathbb{E}^3$ be two (sufficiently smooth) curves of length L parametrized by an arc length s. Suppose that α_1 is a planar curve such that α_1 together with the chord joining its endpoints bounds a convex set in the plane. Let d_1 and d_2 denote the lengths of the chords joining the endpoints of α_1 and α_2, respectively. Assume that*

$$\kappa_1(s) \geq \kappa_2(s), \quad s \in [0, L].$$

Then

$$d_1 \leq d_2,$$

and the equality holds if and only if α_1 and α_2 are congruent.

In the words of S.S. Chern, "if an arc is 'stretched', the distance between its endpoints becomes longer" [2, p. 36]. The curvatures κ_i are assumed in the space sense, that is, non-negative by definition, even for plane curves. This result is known in the literature under the name *Schur's theorem*. Exactly, Schur [4] proved the result when both curvatures agree pointwise (based on ideas of an unpublished work of H.A. Schwarz in 1884). The proof in the general case is due to Schmidt [5]. Some generalizations and extensions can be seen in [6,7] and a historical account of this theorem appears in [7, p. 151]. When the curves are polygons, Schur's theorem is related with the so-called Cauchy's arm lemma [8,9].

In this work we present the Lorentzian version of Schur's theorem for planar curves in the Minkowski plane $\mathbb{E}^2_1$.

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Theorem 2. *Let $\alpha_1, \alpha_2 : [0, L] \to \mathbb{E}_1^2$ be two (sufficiently smooth) spacelike curves of length L parametrized by the arc length s. Suppose that α_1 together with the chord joining its endpoints bounds a convex set. Let d_1 and d_2 denote the lengths of the chords joining the endpoints of α_1 and α_2, respectively. Assume that*

$$\kappa_1(s) \geq |\kappa_2(s)|, \quad s \in [0, L].$$

Then

$$d_1 \geq d_2,$$

and the equality holds if and only if α_1 and α_2 are congruent. The same result holds if both curves are timelike.

Here the length d_i of each chord is measured with the Lorentzian metric, that is, $d_i = |\alpha_i(L) - \alpha_i(0)|$, $i = 1, 2$.

We point out that Schur's theorem cannot be extended for space curves in the Minkowski three-dimensional space $\mathbb{E}_1^3$, even if both curvatures κ_i agree pointwise. We now present a variety of counterexamples. Assume that $\mathbb{E}_1^3$ is the real vector space $\mathbb{R}^3$ with the metric $(+, +, -)$. We consider the (planar) spacelike curves in $\mathbb{E}_1^3$ given by

$$\alpha_1(r; s) = (r\cos(s/r), r\sin(s/r), 0), \qquad \alpha_2(r; s) = (0, r\sinh(s/r), r\cosh(s/r)),$$

with $r > 0$. Both curves are parametrized by the arc length s and the curvatures are $\kappa_i(r; s) = 1/r$, for any s, $i = 1, 2$. Fix L. Then $d_1 = d_1(r) = |\alpha_1(r; L) - \alpha_1(r; 0)| = r\sqrt{2(1 - \cos(L/r))}$ and $d_2 = d_2(r) = |\alpha_2(r; L) - \alpha_2(r; 0)| = r\sqrt{2(\cosh(L/r) - 1)}$.

- By taking the same value r in both curves, and for any $0 < L < 2\pi r$, we have $\kappa_1(r; s) = \kappa_2(r; s)$ for any s and $d_1(r) < d_2(r)$.
- Let $0 < L < \pi$. Then $\kappa_1(1; s) > \kappa_2(2; s)$ and $d_1(r) < d_2(r)$.
- For any $L > 0$, $\kappa_2(1; s) > \kappa_2(2; s)$ and $d_2(1) > d_2(2)$.

This note is organized as follows. In Section 2 we present some preliminaries about the geometry of the Minkowski plane and we show Theorem 2. In Section 3 we present some applications of Schur's theorem.

2. Preliminaries and proof of the result

Let $\mathbb{E}_1^2$ denote the Minkowski plane, that is, the real planar vector $\mathbb{R}^2$ endowed with the metric $\langle , \rangle = dx^2 - dy^2$, where (x, y) are the usual coordinates of $\mathbb{R}^2$. A vector $v \in \mathbb{E}_1^2$ is said to be spacelike if $\langle v, v\rangle > 0$ or $v = 0$, timelike if $\langle v, v\rangle < 0$, and lightlike if $\langle v, v\rangle = 0$ and $v \neq 0$. The length of a vector v is given by $|v| = \sqrt{|\langle v, v\rangle|}$. If $p, q \in \mathbb{E}_1^2$, we define the distance between p and q as $|p - q|$.

It is important to point out that, in contrast to what happens in Euclidean space, in the Minkowski ambient space we cannot define the angle between two vectors, except that both vectors are of timelike type. If $u, v \in \mathbb{E}_1^2$ are two timelike vectors, then $\langle u, v\rangle \neq 0$. In such a case, there is a Cauchy–Schwarz inequality given by

$$|\langle u, v\rangle| \geq |u|\,|v|$$

and the equality holds if and only if u and v are two proportional vectors. In the case where $\langle u, v\rangle < 0$, there exists a unique number $\theta \geq 0$ such that

$$\langle u, v\rangle = -|u|\,|v|\cosh(\theta).$$

This number θ is called the *hyperbolic angle* between u and v.

Given a regular smooth curve $\alpha : I \subset \mathbb{R} \to \mathbb{E}_1^2$, we say that α is spacelike (resp. timelike) if $\alpha'(t)$ is a spacelike (resp. timelike) vector for all $t \in I$, where $\alpha'(t) = d\alpha/dt$. If α is a spacelike curve parametrized by $\alpha(t) = (x(t), y(t))$, then $x'(t)^2 - y'(t)^2 > 0$, and so, $x'(t) \neq 0$ for any t. This means that $x := x(t) : I \to \mathbb{R}$ is a local diffeomorphism, and so, $x : I \to J = x(I)$ is a diffeomorphism between interval of the real line $\mathbb{R}$. Then we can reparametrize α as the graph of a function $y = f(x)$, $x \in J \subset \mathbb{R}$ and $1 - f'(x)^2 > 0$, $x \in J$. In a similar way, any timelike curve in $\mathbb{E}_1^2$ is the graph of a function $x = g(y)$, with $1 - g'(y)^2 > 0$, $y \in J$. In conclusion, there are no *closed* spacelike (or timelike) curves in $\mathbb{E}_1^2$.

The following result will be useful in the proof of Theorem 2.

Lemma 3. *Let $\alpha : I \to \mathbb{E}_1^2$ be a spacelike (resp. timelike) curve. Then for any $s_1, s_2 \in I$, the vector $\alpha(s_2) - \alpha(s_1)$ is spacelike (resp. timelike).*

Proof. We write α as the graph of a function f. By using the mean value theorem, the straight line joining $\alpha(s_1)$ and $\alpha(s_2)$ is parallel to other one that it is tangent at some point $s_3 \in I$. Since $\alpha'(s_3)$ is a spacelike (resp. timelike) vector, then the vector $\alpha(s_2) - \alpha(s_1)$ is also spacelike (resp. timelike). $\square$