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Dynamics of 4D symplectic maps near a double resonance

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ABSTRACT

We study the dynamics of a family of 4D symplectic mappings near a doubly resonant elliptic fixed point. We derive and discuss algebraic properties of the resonances required for the analysis of a Takens type normal form. In particular, we propose a classification of the double resonances adapted to this problem, including cases of both strong and weak resonances.

Around a weak double resonance (a junction of two resonances of two different orders, both being larger than 4) the dynamics can be described in terms of a simple (in general non-integrable) Hamiltonian model. The non-integrability of the normal form is a consequence of the splitting of the invariant manifolds associated with a normally hyperbolic invariant cylinder.

We use a 4D generalisation of the standard map in order to illustrate the difference between a truncated normal form and a full 4D symplectic map. We evaluate numerically the volume of a 4D parallelotope defined by 4 vectors tangent to the stable and unstable manifolds respectively. In good agreement with the general theory this volume is exponentially small with respect to a small parameter and we derive an empirical asymptotic formula which suggests amazing similarity to its 2D analog.

Different numerical studies point out that double resonances play a key role to understand Arnold diffusion. This paper has to be seen, also, as a first step in this direction.

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1. Introduction

It is generally accepted that resonant phenomena play an important role in the theory of Hamiltonian systems and its applications. An important example is the Nekhoroshev theory which provides an upper bound on the rate of Arnold diffusion. In this theory the analysis of dynamics near resonances of higher orders is at the centre of the argument [1,2]. Several numerical studies [3–6] support the fact that multiple resonances should play an important role in an explanation of the Arnold diffusion [7]. It is expected that along a simple resonance, crossed by high or very high order resonances, the main tool is the well-known transition chain mechanism similar to the one originally proposed by Arnold [8]. The main effect of the resonance is to create a "pendulum-like" system and, hence, integrable. Analysis of multiple resonances, even in the simplest approximations, leads to a Hamiltonian system with two or more degrees of freedom, generically non-integrable.

Resonances usually appear in one of the following contexts:

- small perturbations of an integrable system,
- small oscillations near an equilibrium,
- small oscillations near a periodic orbit.

In the last case the dynamics can also be described by a Poincaré map near its fixed point. In each one of these cases, the normal form theory provides a powerful tool for the study of the local dynamics [9].

Each resonance can be characterised by a number of independent resonant relations between the frequencies, this number being called the *multiplicity* of the resonance. In the case of a simple resonance the normal form is integrable, while a double resonance can lead to non-integrability (see e.g. the book [10]). In particular, in a near-integrable system with three or more degrees of freedom the dynamics near a double resonance can be described by a two degrees of freedom Hamiltonian system, which has the form of a sum of a quadratic form in action variables plus a periodic potential [9]. In general this normal form does not contain any perturbation parameter, a fact that substantially complicates its analysis. Haller [10] showed non-integrability of the corresponding normal form in the case when one of the two fundamental resonances has a relatively small amplitude compared to the other one. This situation should generically arise if one of the resonances has a very high order [11,10]. On the other hand this assumption is not expected to be satisfied if both fundamental resonances are of comparable orders.

In this paper we provide a detailed analysis of the normal form near a fixed point of a 4D symplectic map. The fixed point is assumed to be elliptic and with simple eigenvalues, that is, we explicitly avoid the cases of double eigenvalues in 1:1 or 1:-1

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resonance. We recall that small perturbations in the former case lead generically to a pair of simple eigenvalues (it is a structurally stable situation) while might lead to complex instability in the later case, see [12–15] for details. In contrast, the analysis we present extends and completes the original analysis carried out in [16]. In particular, we provide a more detailed classification of the cases with different dynamical behaviour in the normal form.

The paper contains the following results. In Section 2 we provide mostly known results on the classification of the resonances, describe the Takens normal form adapted to the Hamiltonian set up, and analyse the symmetries of the normal form Hamiltonian. In Section 3 we discuss arithmetic properties of the double resonances which can be considered as a two dimensional sub-lattice of \mathbb{Z}^2 . In particular, we prove that all the resonances can be generated by two independent resonances of the lowest orders, which we call minimal generators. These pairs are not unique. For a majority of resonances this non-uniqueness is rather trivial — it originates from the symmetry of the set of resonances with respect to the reflection around the origin. But for some frequencies the nonuniqueness is not trivial and it is reflected in a more complicated structure of a normal form truncated at the order corresponding to the lowest resonant order.

We will also show that for some frequencies two independent resonances of the lowest possible orders may or may not generate all other resonances, and we shall establish sufficient conditions to ensure they do it. The cases when the minimal generators are not unique are discussed in Section 3.4.

In Section 3.2 we provide the complete list of eigenvalues which lead to two independent resonance relations of order 3 or 4. We call them doubly strong resonances, since their normal forms have, generically, no integrable approximation.

In Section 4 we show that a weak double resonance has (in general) a non-integrable normal form. Earlier Kovacic [17] and Haller and Wiggins [18] performed a similar analysis for a nearly integrable system. That analysis required a set of additional hypothesis on the amplitudes of the coefficients in the normal form, which cannot be guaranteed for a generic system. In particular, to ensure such a decay of the coefficients, they obtain a model to describe the dynamics at the junction of the two resonances that requires them to be of quite different order. In this paper, using a similar method, we arrive to more generic conclusions because we take advantage of the properties of the normal form near a fixed point. As a consequence, the (non-integrable) model we obtain from the normal form describes the dynamics in a suitable vicinity around the junction of the resonances provided they are of different order, including the case of similar orders.

Our numerical experiments also have some unusual consequences. Indeed, numerically it can be difficult to distinguish between a very smooth and an analytical invariant surface. On the other hand, it is well known that the general theory of normal hyperbolicity guaranties the existence of invariant manifolds which are smooth, but it is expected that they are not analytic. We provide a strong numerical evidence that a normally hyperbolic invariant cylinder of the truncated normal form is not analytic and has the degree of regularity expected from empirical arguments. The details are placed in Section 4.2. We note that similar experiments can be conducted with a 4D standard-like map used in this paper as a model for the dynamics near a double resonance.

In Section 5 we analyse numerically the difference between an original map and its normal form. We take the 4D standard-like map introduced before as a model and we study the splitting of the invariant manifolds of its hyperbolic fixed point. We find several primary homoclinic points. Our numerical study shows that these points are transversal. Moreover, we show that a suitable volume taken as a measure of transversality becomes exponentially small in the bifurcation parameter. Such an exponentially small behaviour follows indeed from the standard suspension and averaging techniques since the map is near-the-identity [19,20], see related comments in Section 5. Note that as the normal form is given by a Hamiltonian flow, the preservation of the energy of the Hamiltonian implies that such a volume is zero in the normal form case. Furthermore, we found an empirical formula for the asymptotic behaviour of such splitting which resembles the well studied 2D case. This suggests that the methods developed for the study of two dimensional maps can be extended to higher dimensions, although we are not aware of any analytical result in this direction.

2. Resonant normal forms

2.1. Classification of resonances

In this paper we carry out the analysis of the normal forms for a 4-dimensional symplectic map near a totally elliptic fixed point. More precisely, we consider a two-parametric family of 4D symplectic maps F_{δ} which depend on a small enough vector parameter $\delta = (\delta_1, \delta_2)$. We assume that the maps have a fixed point at the origin for all δ

$$F_{\delta}(\mathbf{0}) = \mathbf{0}$$

and $DF_0(\mathbf{0})$ has four non-real eigenvalues on the unit circle $\lambda_1, \bar{\lambda}_1$, $\lambda_2, \bar{\lambda}_2$. Without losing in generality, we may assume that $\text{Im}\lambda_k > 0$. Then there exist $\alpha_k \in (0, \frac{1}{2})$ such that $\lambda_k = \exp(2\pi i\alpha_k), k = 1, 2$. In this paper we will always assume that the eigenvalues are simple, i.e., $\alpha_1 \neq \alpha_2$.

The local dynamics of the map can be described in terms of a normal form which depends on the arithmetic properties of α_k in an essential way. More precisely, the normal form is determined by the set of resonances which form a subgroup $\Gamma \subset \mathbb{Z}^2$ defined

$$\Gamma = \{ (k_1, k_2) : k_1 \alpha_1 + k_2 \alpha_2 = 0 \pmod{1} \}. \tag{2.1}$$

We say that $\mathbf{r} = (k_1, k_2) \in \Gamma$ is a resonance of order $|\mathbf{r}| = |k_1| + |k_2|$. We note that $(k_1, k_2) \in \Gamma$ if and only if

$$\lambda_1^{k_1} \lambda_2^{k_2} = 1. (2.2)$$

Of course (0, 0) is always resonant, and we call it the trivial (or unavoidable) resonance.

The fixed point of F_0 , at the origin, is of one of the following three

1. Non-resonant (Γ is a trivial group).

In this case $\{\alpha_1, \alpha_2, 1\}$ are rationally independent.

2. Simply resonant (Γ is a one-dimensional lattice).

In this case there are two possibilities:

(a) α₁ ∈ ℚ, α₂ ∈ ℝ \ ℚ (or vice versa).
(b) α₁, α₂ ∈ ℝ \ ℚ but {α₁, α₂, 1} are rationally dependent.
3. Doubly resonant (Γ is a two-dimensional lattice).

In this case $\alpha_1, \alpha_2 \in \mathbb{Q}$ and we can write them in the form of

$$\alpha_1 = \frac{p_1}{q_1}$$
 and $\alpha_2 = \frac{p_2}{q_2}$, $p_1, p_2, q_1, q_2 \in \mathbb{N}$.

It is well known that non-resonant and simply resonant normal forms are integrable due to the existence of a continuous family of symmetries. In the case of a double resonance the normal form is expected to be non-integrable and one of our goals is to understand what kind of dynamics can be expected there.

Obviously, both simply resonant and doubly resonant eigenvalues are dense in the torus

$$\mathcal{T} = \{(\lambda_1, \lambda_2) \in \mathbb{C}^2 : |\lambda_1| = |\lambda_2| = 1\}.$$

We note that each resonant relation $\alpha_1 k_1 + \alpha_2 k_2 = k_3$, where $k_1, k_2, k_3 \in \mathbb{Z}$, defines a line on this torus. The Fig. 1 shows all the resonant lines up to the order 12. The picture represents one quarter of the torus. The whole torus can be obtained by reflecting with respect to the coordinate axes.

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