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The Lagrangian–Hamiltonian formalism for higher order field theories

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ABSTRACT

We generalize the Lagrangian–Hamiltonian formalism of Skinner and Rusk to higher order field theories on fiber bundles. As a byproduct we solve the long standing problem of defining, in a coordinate free manner, a Hamiltonian formalism for higher order Lagrangian field theories. Namely, our formalism does only depend on the action functional and, therefore, unlike previously proposed ones, is free from any relevant ambiguity.

0. Introduction

First order Lagrangian mechanics can be generalized to higher order Lagrangian field theory. Moreover, the latter has got a very elegant geometric (and homological) formulation (see, for instance, [1]) on which there is general consensus. On the other hand, it seems that the generalization of Hamiltonian mechanics of Lagrangian systems to higher order field theory presents some more problems. Several answers have been proposed (see, for instance, [2–10] and the references therein) to the question: is there any reasonable, higher order, field theoretic analogue of Hamiltonian mechanics? In our opinion, none of them is satisfactorily natural, especially because of the common emergence of ambiguities due to either the arbitrary choice of a coordinate system [2] or the choice of a Legendre transform [7,8,10]. Namely, the latter seems not to be uniquely definable, except in the case of first order Lagrangian field theories when a satisfactory Hamiltonian formulation can be presented in terms of multisymplectic geometry (see, for instance, [11]—see also [12] for a recent review, and the references therein).

Nevertheless, it is still desirable to have a Hamiltonian formulation of higher order Lagrangian field theories enjoying the same nice properties as Hamiltonian mechanics, which (1) is natural, i.e., is independent of the choice of any structure other than the action functional, (2) gives rise to first order equations of motion, (3) takes advantage of the (pre-)symplectic geometry of the phase space, (4) is a natural starting point for gauge reduction, (5) is a natural starting point for quantization. The relationship between the Euler–Lagrange equations and the Hamilton equations deserves a special mention. The Legendre transform maps injectively solutions of the former to solutions of the latter, but, generically, Hamilton equations

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are not equivalent to Euler–Lagrange ones [11]. However, the difference between the two is a pure gauge and, therefore, it is irrelevant from a physical point of view.

In this paper we achieve the goal of finding a natural (in the above mentioned sense), geometric, higher order, field theoretic analogue of Hamiltonian mechanics of Lagrangian systems in two steps: first, we find a higher order, field theoretic analogue of the Skinner and Rusk "mixed Lagrangian–Hamiltonian" formalism [13–15] (see also [16]), which is rather straightforward (see [17] for a different, finite dimensional approach, to the same problem) and, second, we show that the derived theory "projects to a smaller space" which is naturally interpreted as phase space. Local expressions of the field equations on the phase space are nothing but de Donder equations [2] and, therefore, are naturally interpreted as the higher order, field theoretic, coordinate free analogue of Hamilton equations. A central role is played in the paper by multisymplectic geometry in the form of partial differential (PD, in the following) Hamiltonian system theory, which has been developed in [18].

The paper is divided into nine sections. The first four sections contain reviews of the main aspects of the geometry underlying the paper. They have been included in order to make the paper as self-consistent as possible. The next five sections contain most of the original results.

The first section summarizes the notations and conventions adopted throughout the paper. It also contains references to some differential geometric facts which are often used in the subsequent sections. Finally, in Section 1 we briefly review the Skinner–Rusk formalism [14]. Section 2 is a short review of the geometric theory of partial differential equations (PDEs) (see, for instance, [19]). Section 3 outlines the properties of the main geometric structure of jet spaces and PDEs, the Cartan distribution, and reviews the geometric formulation of the calculus of variations [1]. Section 4 reviews the theory of PD-Hamiltonian systems and their PD-Hamilton equations [18]. Moreover, it contains examples of morphisms of PDEs coming from such theory. These examples are presented here for the first time.

In Section 5 we present the higher order, field theoretic analogue of Skinner–Rusk mixed Lagrangian–Hamiltonian formalism for mechanics. In Section 5 we also discuss the relationship between the field equations in the Lagrangian–Hamiltonian formalism (now on, ELH equations) and the Euler–Lagrange equations. In Section 6 we discuss some natural transformations of the ELH equations. As a byproduct, we prove that they are independent of the choice of a Lagrangian density, in the class of those yielding the same Euler–Lagrange equations, up to isomorphisms. ELH equations are, therefore, as natural as possible. In Section 7 we present our proposal for a Hamiltonian, higher order, field theory. Since we don't use any additional structure other than the ELH equations and the order of a Lagrangian density, we judge our theory satisfactorily natural. Moreover, the associated field equations (HDW equations) are first order and, more specifically, of the PD-Hamilton kind. In Section 8 we study the relationship between the HDW equations and the Euler–Lagrange equations. As a byproduct, we derive a new (and, in our opinion, satisfactorily natural) definition of Legendre transform for higher order, Lagrangian field theories. It is a non-local morphism of the Euler–Lagrange equations into the HDW equations. Finally, in Section 9 we apply the theory to the KdV equation which can be derived from a second order variational principle.

1. Notations, conventions and the Skinner–Rusk formalism

In this section we collect notations and conventions about some general constructions in differential geometry that will be used in the following.

Let N be a smooth manifold. If $L \subset N$ is a submanifold, we denote by $i_L : L \hookrightarrow N$ the inclusion. We denote by $C^\infty(N)$ the $\mathbb{R}$-algebra of smooth, $\mathbb{R}$-valued functions on N. We will always understand a vector field X on N as a derivation $X : C^\infty(N) \longrightarrow C^\infty(N)$. We denote by $\mathrm{D}(N)$ the $C^\infty(N)$-module of vector fields over N, by $\Lambda(M) = \bigoplus_k \Lambda^k(N)$ the graded $\mathbb{R}$-algebra of differential forms over N and by $d : \Lambda(N) \longrightarrow \Lambda(N)$ the de Rham differential. If $F : N_1 \longrightarrow N$ is a smooth map of manifolds, we denote by $F^* : \Lambda(N) \longrightarrow \Lambda(N_1)$ the pull-back via F. We will understand everywhere the wedge product $\wedge$ of differential forms, i.e., for $\omega, \omega_1 \in \Lambda(N)$, we will write $\omega\omega_1$ instead of $\omega \wedge \omega_1$.

Let $\alpha : A \longrightarrow N$ be an affine bundle (for instance, a vector bundle) and $F : N_1 \longrightarrow N$ a smooth map of manifolds. Let $\mathscr{A}$ be the affine space of smooth sections of α. For $a \in \mathscr{A}$ and $x \in N$ we put, sometimes, $a_x := a(x)$. The affine bundle on N_1 induced by α via F will be denoted by $F^\circ(\alpha) : F^\circ(A) \longrightarrow N$:

$$\begin{array}{ccc} F^\circ(A) & \longrightarrow & A \\ {\scriptstyle F^\circ(\alpha)}\downarrow & & \downarrow{\scriptstyle \alpha} \\ N_1 & \xrightarrow{F} & N \end{array} \quad ,$$

and the space of its sections by $F^\circ(\mathscr{A})$. For any section $a \in \mathscr{A}$ there exists a unique section, which we denote by $F^\circ(a) \in F^\circ(\mathscr{A})$, such that the diagram

$$\begin{array}{ccc} F^\circ(A) & \longrightarrow & A \\ {\scriptstyle F^\circ(a)}\uparrow & & \uparrow{\scriptstyle a} \\ N_1 & \xrightarrow{F} & N \end{array}$$