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Reverse draining of a magnetic soap film — Analysis and simulation of a thin film equation with non-uniform forcing

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ABSTRACT

We analyze and classify equilibrium solutions of the one-dimensional thin film equation with no-flux boundary conditions and in the presence of a spatially dependent external forcing. We prove theorems that shed light on the nature of these equilibrium solutions, guarantee their validity, and describe how they depend on the properties of the external forcing. We then apply these results to the reverse draining of a one-dimensional magnetic soap film subject to an external non-uniform magnetic field. Numerical simulations illustrate the convergence of the solutions towards equilibrium configurations. We then present bifurcation diagrams for steady state solutions. We find that multiple stable equilibrium solutions exist for fixed parameters, and uncover a rich bifurcation structure to these solutions, demonstrating the complexity hidden in a relatively simple looking evolution equation. Finally, we provide a simulation describing how numerical solutions traverse the bifurcation diagram, as the amplitude of the forcing is slowly increased and then decreased.

1. Introduction

As a vertical soap film drains under gravity, a growing region of very thin film, termed black film, forms at the top. This process has been studied by a number of authors, and for a variety of reasons, including the capture of film properties, and the understanding of foams [1,2]. Much of the interest, with contributions dating as far back as Newton [3], seems to arise from the complexity underlying the process. The physical mechanisms behind a draining film have been analyzed and put in concrete mathematical terms in more recent times, largely beginning with the work of Mysels et al. in the 1950s [4]. In particular, they introduced the concept of marginal regeneration to explain the formation of thin regions of film along the film borders. Since then, multiple authors have added to the topic (e.g, [5–7]), both experimentally and theoretically.

The process of marginal regeneration, in particular the mechanisms responsible for the creation of black film at the film borders and the forces that drive the motion of the thin film once it is created, is still somewhat controversial. For instance, in [7] the authors suggest that the formation of thin film is a surface tension effect, contrary to the original explanation given by Mysels et al. In [8] the commonly held understanding that the thin film's subsequent motion is solely due to gravity is called into question. The dynamics of a draining film come from the competition between viscous, capillary, and gravitational forces, along with surface tension effects and potentially complex interactions with the film boundary. That some controversy remains today confirms that these systems which may at first appear relatively simple are actually quite complex.

One way to better understand a draining film and to capture the effects of different parameters is to add a controllable component to the system. In [9], Elias et al. explored the physical properties of soap films by adding an aqueous suspension of magnetic nanoparticles to an ordinary soap solution, thus forming a magnetic soap solution. This added magnetic dimension was treated as a macroscopic force which they could control by subjecting the film to varying magnetic fields. By placing a vertical draining film in a uniform magnetic field, they found that they could speed up or slow down the draining process based on the orientation of the magnetic field. More recently, Moulton and Pelesko [8] presented a similar experimental setup with a magnetic soap film, but with a key difference: they subjected a draining film to much stronger and non-uniform magnetic fields, by placing strong bar magnets above the vertical film. With this setup, they found that with a strong enough magnet, the film would flow upwards against gravity, with thin black film forming at the bottom, a process termed reverse draining.

In [8], a first model is suggested for the draining film under the presence of a non-uniform magnetic field. Numerical simulations demonstrate qualitative agreement with experimental observation, but a rigorous analysis of the system is not given. We provide such an analysis in this paper.

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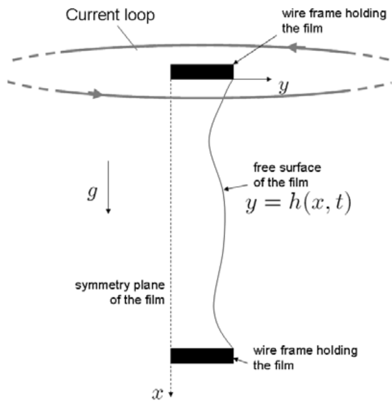

Fig. 1. Setup for the draining magnetic film system on which the model is based. See text for details.

The system we study is depicted in Fig. 1. The film is described by the function $y = h(x,t)$, which is the half-thickness of the film, assuming a reflection symmetry about the center line. The full film is envisioned by extending the profile in the transverse z direction — that is, the film is a sheet independent of z and with reflectional symmetry about $y = 0$. The film is assumed to be tangentially immobile, meaning that there is a zero tangential velocity along the free surface. The magnet is modeled by a current loop placed directly above the film, with the film sitting on the axis of the loop. The radius of the current loop is assumed to be much larger than the width of the film in the transverse z direction. This results in a non-uniform magnetic force acting only in the vertical x direction to first order, and models the effect that magnetic particles are primarily being pulled upwards, and that particles closer to the magnet will feel a stronger pull. To be physically realistic, a magnetic force acting solely in the vertical direction to first order is only compatible with the assumption that the film is uniform in the transverse z direction if the width of the film is small compared to the radius of the current loop, which may or may not agree with experiment. However, such a magnetic field is necessary in order to formulate a two-dimensional first model, which is certainly desirable from an analytical perspective. The model exploits several other simplifications, such as the absence of surfactant transport, in order to focus on the basic mechanism and effect of the magnetic field. Moreover, only macroscopic effects of the magnetic field are taken into account, and magnetic particle concentration is not considered. Nevertheless, we will see that this model has a very rich set of solutions, and is also capable of capturing the reverse draining effect observed in experiments.

An evolution equation for the film can be derived using lubrication theory, under the assumption of small aspect ratio of the thickness of the film to its length. A comprehensive review of lubrication theory and its application to various thin film systems is found in [10]. For the setup studied here, the fluid is assumed incompressible, with constant density and viscosity. Symmetry conditions are employed at the center line $y = 0$, while the assumption of a tangentially immobile film provides the boundary condition of zero velocity in the vertical x direction at the free surface. The effects of gravity, viscosity, and surface tension are included; a magnetic body force term also appears as described by the theory of ferrohydrodynamics [11]. Starting from the Navier–Stokes equations, and following the standard procedures, the following evolution equation for the film is obtained:

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\left(\frac{h^3}{3}\left[\sigma h_{xxx} + 1 + \lambda f(x)\right]\right) = 0. \tag{1}$$

There are two dimensionless parameters: σ is an inverse Capillary number characterizing surface tension, while λ is a ratio of the relative forces of the magnetic field and gravity. Gravity has been scaled to be the factor of 1, and

$$f(x) = \frac{-3\eta^2 x}{(1+\eta^2 x^2)^4} \tag{2}$$

is the magnetic forcing function for the non-uniform magnetic field, where η is the ratio of the radius of the current loop to the length of the film. For a detailed derivation of Eq. (1) and the magnetic term described by Eq. (2), see [8]. Note that if we remove the term $\lambda f(x)$, the remaining equation describes the evolution of a thin film under the external action of gravity only. This reduced equation, or some close variant of it, appears in a number of previous studies. For instance, in [12], this exact equation is studied, albeit with different boundary conditions than we will employ here. Similar models are also developed in [13,14]; in these studies the analysis is complicated by the addition of an equation for surfactant transport coupled to the evolution equation. In [15], the effect of an electric field on a thin film is explored — the evolution equation presented takes a similar form, and is coupled to an equation for the electric potential in the region of space outside the film.

Eq. (1) falls under the category of fourth order degenerate diffusion equations, which arise through the lubrication approximation. Several papers have analyzed these types of equations in a general setting, examples include [16,17]. The issues addressed in these papers appear in the present work as well. Note however, that in these studies the evolution equation analyzed is autonomous. The addition of the non-autonomous function $f(x)$ to the thin film equation greatly changes the film behavior and also complicates some standard techniques of analysis; the effect of this added function is a key element of our analysis.

The domain of the film is $0 \le x \le 1$, where $x = 0$ corresponds to the top of the film and the location of the current loop, and $x = 1$ is the bottom of the film. Note that the term inside the x derivative is the velocity flux $Q(x,t)$ over a horizontal cross section; that is

$$Q(x,t) = \frac{h^3}{3}\left[\sigma h_{xxx} + 1 + \lambda f(x)\right].$$

There are several options for boundary conditions, depending on the specifics of the experiment. In [8], experiments consist of a film formed over an isolated rectangular frame. Hence, the natural boundary conditions, which we use in this paper, are

$$h(0,t) = h(1,t) = 1, \qquad Q(0,t) = Q(1,t) = 0. \tag{3}$$

Physically, the assumption is that the film is pinned to the frame and there is no flux across the frame at either end. Note that the no-flux condition implies a volume conservation.

Our objective in this paper is to analyze the system given by Eq. (1) with boundary conditions (3). We begin in Section 2 with a numerical investigation. A key characteristic that emerges is that all solutions approach steady state equilibrium profiles. In Section 3, we study analytically these equilibrium solutions. An interesting aspect is that piecewise equilibrium solutions may be constructed, which by their nature contain singularities in the third derivative. We prove several theorems regarding the construction and validity of equilibrium solutions, and how the shape of the forcing function $f(x)$ dictates the shape of equilibrium profiles. This analysis is conducted for an arbitrary forcing function. In Section 4, we numerically explore the convergence of different solution profiles to a steady state profile as well as their stability. We also illustrate the rich structure of the solution set, and consider the evolution and bifurcation of solutions as the magnetic field strength is altered.

2. Numerical solution

We begin with numerical simulations of the system (1)–(3), using the method of lines. Motivated by the zero flux boundary