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Symplectic maps of complex domains into complex space forms☆

Andrea Loi*, Fabio Zuddas

Dipartimento di Matematica e Informatica, Università di Cagliari, Italy

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Abstract

Let $M \subset \mathbb{C}^n$ be a complex domain of $\mathbb{C}^n$ endowed with a rotation invariant Kähler form $\omega_\Phi = \frac{i}{2}\partial\bar{\partial}\Phi$. In this paper we describe sufficient conditions on the Kähler potential Φ for (M, ω_Φ) to admit a symplectic embedding (explicitly described in terms of Φ) into a complex space form of the same dimension of M. In particular we also provide conditions on Φ for (M, ω_Φ) to admit global symplectic coordinates. As an application of our results we prove that each of the Ricci-flat (but not flat) Kähler forms on $\mathbb{C}^2$ constructed by LeBrun in [C. LeBrun, Complete Ricci-flat Kähler metrics on $\mathbb{C}^n$ need not be flat, in: Proceedings of Symposia in Pure Mathematics, vol. 52, 1991, pp. 297–304. Part 2] admits explicitly computable global symplectic coordinates.

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1. Introduction and statements of the main results

Let (M, ω) and (S, Ω) be two symplectic manifolds of dimension $2n$ and $2N$, $n \leq N$, respectively. Then, one has the following natural and fundamental question.

Question 1. Under which conditions does there exist a symplectic embedding $\Psi : (M, \omega) \to (S, \Omega)$, namely a smooth embedding $\Psi : M \to S$ satisfying $\Psi^*(\Omega) = \omega$?

Theorems A–D below give a topological answer to the previous question when Ω is the Kähler form of an N-dimensional complex space form S, namely (S, Ω) is either the complex Euclidean space $(\mathbb{C}^N, \omega_0)$, the complex hyperbolic space $(\mathbb{C}H^N, \omega_{\mathrm{hyp}})$ or the complex projective space $(\mathbb{C}P^N, \omega_{FS})$ (see below for the definition of the symplectic (Kähler) forms ω_0, ω_{hyp} and ω_{FS}). Indeed these theorems are consequences of Gromov's h-principle [12] (see also Chapter 12 in [9] for a beautiful description of Gromov's work).

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* Corresponding author.
E-mail addresses: loi@unica.it (A. Loi), fzuddas@unica.it (F. Zuddas).

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Theorem A (*Gromov [12], see also [10]*). *Let (M, ω) be a contractible symplectic manifold. Then there exist a nonnegative integer N and a symplectic embedding $\Psi : (M, \omega) \to (\mathbb{C}^N, \omega_0)$, where $\omega_0 = \sum_{j=1}^N \mathrm{d}x_j \wedge \mathrm{d}y_j$ denotes the standard symplectic form on $\mathbb{C}^N = \mathbb{R}^{2N}$.*

This was further generalized by Popov as follows.

Theorem B (*Popov [20]*). *Let (M, ω) be a symplectic manifold. Assume ω is exact, namely $\omega = d\alpha$, for a 1-form α. Then there exist a nonnegative integer N and a symplectic embedding $\Psi : (M, \omega) \to (\mathbb{C}^N, \omega_0)$.*

Observe that the complex hyperbolic space $(\mathbb{C}H^N, \omega_{\mathrm{hyp}})$, namely the unit ball $\mathbb{C}H^N = \{z = (z_1, \ldots, z_N) \in \mathbb{C}^N \mid \sum_{j=1}^N |z_j|^2 < 1\}$ in $\mathbb{C}^N$ endowed with the hyperbolic form $\omega_{\mathrm{hyp}} = -\frac{i}{2}\partial\bar\partial \log(1 - \sum_{j=1}^N |z_j|^2)$ is globally symplectomorphic to $(\mathbb{C}^N, \omega_0)$ (see (20) in Lemma 2.2) hence Theorem B immediately implies

Theorem C. *Let (M, ω) be a symplectic manifold. Assume ω is exact. Then there exist a nonnegative integer N and a symplectic embedding $\Psi : (M, \omega) \to (\mathbb{C}H^N, \omega_{\mathrm{hyp}})$.*

The following theorem, further generalized by Popov [20] to the noncompact case, deals with the complex projective $\mathbb{C}P^N$, equipped with the Fubini–Study form ω_{FS}. Recall that if $Z_0, \ldots, Z_N$ denote the homogeneous coordinates on $\mathbb{C}P^N$, then, in the affine chart $Z_0 \neq 0$ endowed with coordinates $z_j = \frac{Z_j}{Z_0}$, $j = 1, \ldots, N$, the Fubini–Study form reads as

$$\omega_{FS} = \frac{i}{2}\partial\bar\partial \log\left(1 + \sum_{j=1}^N |z_j|^2\right).$$

Theorem D (*Gromov [10], see also Tischler [22]*). *Let (M, ω) be a compact symplectic manifold such that ω is integral. Then there exist a nonnegative integer N and a symplectic embedding $\Psi : (M, \omega) \to (\mathbb{C}P^N, \omega_{FS})$.*

At this point a natural problem is that to find the smallest dimension of the complex space form where a given symplectic manifold (M, ω) can be symplectically embedded. In particular one can study the case of equidimensional symplectic maps, as expressed by the following interesting question.

Question 2. Given a $2n$-dimensional symplectic manifold (M, ω) under which conditions does there exist a symplectic embedding Ψ of (M, ω) into $(\mathbb{C}^n, \omega_0)$ or $(\mathbb{C}P^n, \omega_{FS})$?

Notice that locally there are no obstructions to the existence of such Ψ. Indeed, by a well-known theorem of Darboux for every point $p \in M$ there exist a neighbourhood U of p and an embedding $\Psi : U \to \mathbb{R}^{2n} = \mathbb{C}^n$ such that $\Psi^*(\omega_0) = \omega$. In order to get a local embedding into $(\mathbb{C}P^n, \omega_{FS})$ we can assume (by shrinking U if necessary) that $\Psi(U) \subset \mathbb{C}H^n$. Therefore $f \circ \Psi : U \to (\mathbb{C}^n, \omega_{FS}) \subset (\mathbb{C}P^n, \omega_{FS})$, with f given by Lemma 2.2 below, is the desired embedding satisfying $(f \circ \Psi)^*(\omega_{FS}) = \Psi^*(\omega_0) = \omega$. Observe also that Darboux's theorem is a special case of the following

Theorem E (*Gromov [13]*). *A $2n$-dimensional symplectic manifold (M, ω) admits a symplectic immersion into $(\mathbb{C}^n, \omega_0)$ if and only if the following three conditions are satisfied: a) M is open, b) the form ω is exact, c) the tangent bundle (TM, ω) is a trivial $Sp(2n)$-bundle. (Observe that a), b), c) are satisfied if M is contractible).*

It is worth pointing out that the previous theorem is not of any help in order to attack Question 2 due to the existence of exotic symplectic structures on $\mathbb{R}^{2n}$ (cfr. [11]). (We refer the reader to [1] for an explicit construction of a 4-dimensional symplectic manifold diffeomorphic to $\mathbb{R}^4$ which cannot be symplectically embedded in $(\mathbb{R}^4, \omega_0)$).

In the case when our symplectic manifold (M, ω) is a Kähler manifold, with associated Kähler metric g, one can try to impose Riemannian or holomorphic conditions to answer the previous question. From the Riemannian point of view the only complete and known result (to the authors' knowledge) is the following global version of Darboux's theorem.

Theorem F (*McDuff [19]*). *Let (M, g) be a simply-connected and complete n-dimensional Kähler manifold of nonpositive sectional curvature. Then there exists a diffeomorphism $\Psi : M \to \mathbb{R}^{2n}$ such that $\Psi^*(\omega_0) = \omega$.*