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Non-Hamiltonian systems separable by Hamilton–Jacobi method

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Abstract

We show that with every separable classical Stäckel system of Benenti type on a Riemannian space one can associate, by a proper deformation of the metric tensor, a multi-parameter family of non-Hamiltonian systems on the same space, sharing the same trajectories and related to the seed system by appropriate reciprocal transformations. These systems are known as bi-cofactor systems and are integrable in quadratures as the seed Hamiltonian system is. We show that with each class of bi-cofactor systems a pair of separation curves can be related. We also investigate the conditions under which a given flat bi-cofactor system can be deformed to a family of geodesically equivalent flat bi-cofactor systems.

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1. Introduction

A significant progress in the geometric separability theory for the classical Hamiltonian systems separable by Hamilton–Jacobi method has been achieved in recent years (see for example [1–4]). Among other things a new class of non-Hamiltonian–Newton systems was introduced [5,6]. These systems were shown to have very interesting geometric properties when considered as systems on Riemannian spaces [7,8] (see also [9]). In [10] we showed that they can be separated by the Hamilton–Jacobi method after certain reparametrization of the evolution parameter (see also [11]). Originally these systems were called quasi-Lagrangian systems. In the present literature they are called bi-cofactor systems or cofactor-pair systems. In [12] it was further shown that each bi-cofactor system is *geodesically equivalent* (in the classical sense of Levi-Civita [13]) to some separable Lagrangian system which means that it has the same trajectories on the underlying configuration manifold as the Lagrangian system only traversed with a different speed and moreover that the metric tensors associated with both systems are equivalent i.e. have the same geodesics (considered as unparametrized curves). In the same paper one can also find a thorough geometric theory of bi-cofactor systems on an arbitrary pseudo-Riemannian space.

In the present paper we demonstrate on the level of differential equations the geodesic equivalence properties of cofactor and bi-cofactor systems expressed by an appropriate class of reciprocal transformations (for definition and properties of reciprocal transformations for finite-dimensional integrable systems see [14]). We clarify and

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systematize their bi-quasi-Hamiltonian formulation on the phase space. We show explicitly that a bi-cofactor system is geodesically equivalent to two different separable Hamiltonian systems of Benenti type and we show explicitly the transformation between all geometric structures associated with these two Benenti systems and the original bi-cofactor system. We further demonstrate that with each bi-cofactor system one can relate two different separation curves and we find a map between these curves. From this point of view we therefore show that with each pair of separation curves that are related through the above-mentioned map we can associate a whole class of geodesically equivalent bi-cofactor systems. Every such class contains at least two separable Hamiltonian systems and on the phase space all the members of a given class are related by a composition of an appropriate noncanonical transformation and a reciprocal transformation. Further, we investigate geodesically equivalent families of flat (in the sense of the underlying metric tensor) cofactor systems and find a sufficient condition for a so-called J-tensor to generate from any given flat bi-cofactor system a multi-parameter family of flat bi-cofactor systems. Finally, we illustrate our considerations by presenting a thorough example of the class of separable bi-cofactor systems geodesically equivalent to the Henon–Heiles system and then specify this example to the flat case.

2. Cofactor systems

Let us consider the following Newton system

$$\frac{\mathrm{d}^2q^i}{\mathrm{d}t^2} + \Gamma^i_{jk}\frac{\mathrm{d}q^j}{\mathrm{d}t}\frac{\mathrm{d}q^k}{\mathrm{d}t} = F^i, \qquad i = 1, \dots, n, \tag{1}$$

where q^i are some coordinates on an n-dimensional pseudo-Riemannian manifold Q endowed with a metric tensor $g = (g_{ij})$ and where $F = (F^i)$ is a vector field on Q representing the force which we assume to be time- and velocity-independent. Here and in what follows we use the Einstein summation convention if not stated otherwise. The functions Γ^i_{jk} are the Christoffel symbols of the Levi-Civita connection associated with the metric tensor g and if all Γ^i_{jk} are zero we call the system (1) a *flat Newton system*. In case that $F = 0$ (1) is the equation of geodesic motion on Q and the variable t becomes an affine parameter of geodesic lines.

If the force F is conservative (potential) i.e. if

$$F = -\nabla V = -G\mathrm{d}V, \tag{2}$$

where $G = g^{-1}$ is the contravariant form of the metric tensor g and where $V = V(q)$ is a potential function, then (1) is equivalent to the Lagrangian system

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial v^i} - \frac{\partial \mathcal{L}}{\partial q^i} = 0, \quad v^i = \frac{\mathrm{d}}{\mathrm{d}t}q^i, i = 1, \dots, n \tag{3}$$

on the tangent bundle TQ endowed with coordinates $(q, v) = (q^1, \dots q^n, v^1, \dots, v^n)$, where $\mathcal{L} = \frac{1}{2}g_{ij}(q)v^iv^j - V(q)$ is a Lagrangian of the system. By the Legendre map $p_i = g_{ij}v^j$, the system (3) is transformed to the Hamiltonian dynamical system

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}\begin{pmatrix} \dfrac{\partial H}{\partial q} \\ \dfrac{\partial H}{\partial p} \end{pmatrix} = \Pi_c\, \mathrm{d}H \tag{4}$$

on the cotangent bundle T^*Q endowed with coordinates $(q, p) = (q^i, p_j)$ where $H = \frac{1}{2}G^{ij}(q)p_ip_j + V(q)$ is the Hamiltonian of the system, Π_c is the canonical Poisson tensor and $\mathrm{d}H$ is the differential of H.

We will now remind the notion of a J-tensor.

Definition 1. A $(1, 1)$-tensor $\mathbf{J} = (J^i_j)$ on Q is called a J-tensor associated with the metric g or $G = g^{-1}$ (we often write that $\mathbf{J}$ is a J_G-tensor when emphasizing the underlying metric) if its contravariant form $J^{ij} = J^i_kG^{kj}$ is a symmetric $(2, 0)$-tensor and if $\mathbf{J}$ itself satisfies the following *characteristic equation*

$$\nabla_h J^i_j = \left(\alpha_j\delta^i_h + \alpha^i g_{jh}\right), \tag{5}$$

where ∇_h is the covariant derivative associated with the metric g and where α_i is some 1-form.