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ScienceDirect
Journal of Geometry and Physics 58 (2008) 654–669
JOURNAL OF GEOMETRY AND PHYSICS
www.elsevier.com/locate/jgp

Gromov–Witten invariants of Fano hypersurfaces, revisited

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Received 15 October 2007; received in revised form 21 December 2007; accepted 21 January 2008
Available online 10 February 2008

Abstract

The goal of this paper is to give an efficient computation of the genus zero three-point Gromov–Witten invariants of Fano hypersurfaces, starting from the Picard–Fuchs equation. This simplifies and to some extent explains the original computations of Jinzenji. The method involves solving a gauge-theoretic differential equation, and our main result is that this equation has a unique solution.

MSC: 53D45

Keywords: Quantum cohomology; $\mathcal{D}$-module; Birkhoff factorization

1. Introduction

Gromov–Witten invariants compute "numbers of pseudo-holomorphic curves" in a symplectic manifold. They are rigorously defined as integrals on moduli spaces of stable maps. Therefore it is difficult to calculate Gromov–Witten invariants directly from the definition.

An alternative method of computation comes from mirror symmetry. Although the mirror symmetry phenomenon is not yet completely understood, it suggests that Gromov–Witten invariants can be computed in terms of coefficients of power series solutions of certain differential equations. The most well known example is the quintic hypersurface in $\mathbb{C}P^4$; this is a Calabi–Yau 3-fold. Fano hypersurfaces are more elementary from the point of view of Gromov–Witten invariants, and it was established by Givental that certain Gromov–Witten invariants in this case are determined by the "Picard–Fuchs equation". The Picard–Fuchs equation of the quintic hypersurfaces in $\mathbb{C}P^4$ is $\left(\partial^4 - 5e^t(5\partial+4)(5\partial+3)(5\partial+2)(5\partial+1)\right)\psi(t) = 0$.

A hypersurface M_N^k of degree k in $\mathbb{C}P^{N-1}$ is Fano if and only if $N > k$, and the Picard–Fuchs equation is

$$\left(\partial^{N-1} - ke^t(k\partial+(k-1))\dots(k\partial+2)(k\partial+1)\right)\psi(t) = 0.$$

Before Givental's work, partial results on the quantum cohomology of Fano hypersurfaces had been obtained by Collino and Jinzenji [4] and Beauville [2]. Subsequently, Jinzenji [8] observed that a simple ansatz leads to the correct Gromov–Witten invariants and he obtained complicated but explicit formulae from this ansatz.

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0393-0440/$ - see front matter
doi:10.1016/j.geomphys.2008.01.008

The aim of this paper is to calculate the three-point Gromov–Witten invariants of a Fano hypersurface by using the method of [1,7] primarily when $N - k \geq 2$ (which we assume unless stated otherwise). In this method, the flat connection associated with the $\mathcal{D}$-module $\mathcal{D}/(\mathrm{PF})$ is "normalized" by applying the Birkhoff factorization. We shall show (as a consequence of Givental's work) that this method produces the correct three-point Gromov–Witten invariants.

The algorithm for the computation of three-point Gromov–Witten invariants from the quantum differential equations was introduced in [7], and applied to flag manifolds in [1], and our treatment of hypersurfaces is broadly similar. However, there are some special features in this case which make a separate discussion worthwhile. First, the differential equations in this case are o.d.e., rather than p.d.e., hence the integrability condition plays no role. Second, the o.d.e. which appears in the Birkhoff factorization can be integrated very explicitly, and this leads to purely algebraic formulae (whereas the algorithm in [1] required the solution of large systems of p.d.e.).

Computationally, our method is similar to Jinzenji's method, but considerably simpler. In Section 2 we review the cohomology algebra of hypersurfaces of the complex projective spaces. In Section 3 we discuss the Gromov–Witten invariants and the Dubrovin connection. The quantum differential system and Jinzenji's method are discussed in Section 4. In Section 5, we explain the loop group method and we compute a flat connection from a $\mathcal{D}$-module which is related to the quantum differential system. In Section 6, we discuss relations between families of connection 1-forms and $\mathcal{D}$-modules. The "adapted" gauge group is the most important object. In Section 7, we explain Jinzenji's results from our viewpoint and prove that our results agree with Jinzenji's results. We also prove that our results produce the Gromov–Witten invariants.

2. Hypersurfaces in the complex projective spaces

If we consider the hyperplane

$$H = \left\{[z_0, \ldots, z_{N-1}] \in \mathbb{C}P^{N-1} \mid z_0 = 0\right\}$$

of $\mathbb{C}P^{N-1}$ as a smooth divisor, the line bundle $\mathcal{O}(1)$ over $\mathbb{C}P^{N-1}$ can be obtained from the divisor in the general theory of complex geometry. The line bundle $\mathcal{O}(1)$ is called the hyperplane bundle. The first Chern class b of the hyperplane bundle $\mathcal{O}(1)$ generates the cohomology algebra $H^*(\mathbb{C}P^{N-1};\mathbb{C})$ of $\mathbb{C}P^{N-1}$. The tensor product of k copies of the line bundle $\mathcal{O}(1)$ is denoted by $\mathcal{O}(k)$. The zero locus of a holomorphic section of the line bundle $\mathcal{O}(k)$ is called a hypersurface of degree k in $\mathbb{C}P^{N-1}$ and the zero locus is denoted by M_N^k.

The $\mathbb{C}$-linear space $H^\sharp(M_N^k)$ of all pullbacks of cohomology classes via the inclusion map $\iota : M_N^k \to \mathbb{C}P^{N-1}$ is a subalgebra of the cohomology algebra of M_N^k. The subalgebra $H^\sharp(M_N^k)$ is generated by the pullback of the cohomology class b. The pullback is also denoted by b. Let b_i $(i = 1, \ldots, N-2)$ be a cup product of i copies of b and $b_0 = 1$. The vectors $b_0, \ldots, b_{N-2}$ form a $\mathbb{C}$-basis of the subalgebra $H^\sharp(M_N^k)$.

We assume that $N \geq 5$. Under this assumption, the Lefschetz theorem implies that the homomorphism $\iota_* : H_2(M_N^k;\mathbb{Z}) \to H_2(\mathbb{C}P^{N-1};\mathbb{Z}) = \mathbb{Z}[\mathbb{C}P^1]$ induced by the inclusion is an isomorphism. Taking the generator A of $H_2(M_N^k;\mathbb{Z})$ with $\iota_* A = [\mathbb{C}P^1]$, we identify the second homology group $H_2(M_N^k;\mathbb{Z})$ with $\mathbb{Z}$ via the isomorphism.

There are two nondegenerate bilinear pairings over $\mathbb{C}$; one is the Kronecker pairing (the usual pairing)

$$\langle\ ,\ \rangle : H^2(M_N^k;\mathbb{C}) \otimes H_2(M_N^k;\mathbb{C}) \to \mathbb{C}; \quad \langle x, d\rangle = \int_d x$$

and the other is the Poincaré pairing

$$(\ ,\ ) : H^\sharp(M_N^k) \otimes H^\sharp(M_N^k) \to \mathbb{C}; \quad (x, y) = \int_{M_N^k} x \cup y.$$

Note that $g_{\mu\nu} := (b_\mu, b_\nu) = k\delta_{N-2}^{\mu+\nu}$.

3. Gromov–Witten invariants

The subalgebra $H^\sharp(M_N^k)$ is a Frobenius algebra in the following sense: