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Renormalised Chern–Weil forms associated with families of Dirac operators

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Abstract

We provide local expressions for Chern–Weil type forms built from superconnections associated with families of Dirac operators previously investigated in [S. Scott, Zeta–Chern forms and the local family index theorem, Trans. Amer. Math. Soc. (in press). arXiv: math.DG/0406294] and later in [S. Paycha, S. Scott, Chern–Weil forms associated with superconnections, in: B. Booss-Bavnbeck, S. Klimek, M. Lesch, W. Zhang (Eds.), Analysis, Geometry and Topology of Elliptic Operators, World Scientific, 2006].

When the underlying fibration of manifolds is trivial, the even degree forms can be interpreted as renormalised Chern–Weil forms in as far as they coincide with regularised Chern–Weil forms up to residue correction terms. Similarly, a new formula for the curvature of the local fermionic vacuum line bundles is derived using a residue correction term added to the naive curvature formula.

We interpret the odd degree Chern–Weil type forms built from superconnections as Wodzicki residues and establish a transgression formula along the lines of known transgression formulae for η-forms.

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0. Introduction

The Chern–Weil formalism in finite dimensions assigns to a connection ∇ on a principal bundle $P \to B$ over a manifold B a form $f(\nabla)$ on B with values in the adjoint bundle Ad P:

$$\begin{aligned} f &: \mathcal{C}(P) \to \Omega(B, \mathrm{Ad}\, P) \\ \nabla &\mapsto f(\nabla), \end{aligned}$$

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which is closed with de Rham cohomology class independent of the choice of connection. Here $\mathcal{C}(P)$ is the space of connections on P and $\Omega(B, W)$ the space of differential forms with values in a vector bundle W over B.

In the context of Ψdo-bundles – i.e. bundles with structure group the group of zero-order invertible pseudodifferential operators $C\ell_0^*$ – the trace used in finite dimensions to build maps $f_j(\nabla) = \mathrm{tr}(\nabla^{2j})$ can be replaced by two[1] natural traces on the algebra $C\ell_0$ of zero-order pseudodifferential operators, namely the Wodzicki residue and the leading symbol trace. Such constructions were investigated in [17] and lead to maps which project down to quotient connections $\bar{\nabla}$ on the quotient bundle $\bar{P}$ with structure group $C\ell_0^*/(1 + C\ell_{-\infty})^*$ where $C\ell_{-\infty}$ is the algebra of smoothing operators and $(1 + C\ell_{-\infty})^*$ the group of invertibles. In other words, they project down to maps:

$$\begin{array}{rcl} \bar{f} : \mathcal{C}(\bar{P}) & \to & \Omega(B, \mathrm{Ad}\, \bar{P}) \\ \nabla & \mapsto & \bar{f}(\bar{\nabla}). \end{array}$$

We call such maps *local* in as far as they are insensitive "to smoothing perturbations".

In contrast, on a principal bundle with structure group $(1+C\ell_{-\infty})^* \subset C\ell_0^*$ one can mimic the ordinary Chern–Weil construction to build Chern classes using the ordinary trace on $C\ell_{-\infty}$. We are concerned in this paper with possible extensions of these Chern forms to Ψdo-bundles. Since the ordinary trace on $C\ell_{-\infty}$ extends to linear functionals on $C\ell_0$ obtained from regularised (or weighted) traces, one might want to try to extend the ordinary Chern–Weil constructions to Ψdo-bundles using these regularised traces. Such issues were addressed in [17]; the fact that regularised traces do not yield genuine traces gives rise to obstructions to carrying out the Chern–Weil construction since the regularised Chern forms obtained from regularised traces are not closed. However, it is useful to keep in mind that the obstruction to their closedness can be expressed in terms of local maps in the above sense.

In this paper, we discuss ways to "renormalise" the regularised Chern forms by adding to them local terms in order to turn them into closed forms with de Rham classes independent of the connection. To do so, we compare them with Chern forms previously investigated in [22] and later [19], which are built from superconnections; in some cases they differ by local expressions so that a renormalisation procedure can indeed be carried out adding local counterterms. More precisely, letting (say in the $\mathbb{Z}_2$-graded case) $\mathbb{A} = D + \nabla$ be a superconnection associated with a Dirac operator D, then the expression

$$\mathrm{tr}^{D^2}(\nabla^{2j}) - \mathrm{tr}^{\mathbb{A}^2}(\mathbb{A}^{2j})_{[2j]}$$

(which compares the naive infinite dimensional analogue $\mathrm{tr}^{D^2}(\nabla^{2j})$ of the finite dimensional Chern form $\mathrm{tr}(\nabla^{2j})$ and the closed form $\mathrm{tr}^{\mathbb{A}^2}(\mathbb{A}^{2j})_{[2j]}$ built from the super connection) is local in the above sense. Here $\mathrm{tr}^{D^2}(B) := \mathrm{fp}_{z=0}\mathrm{TR}(B(D^2 + \pi_D)^{-z})$ is the D^2-weighted (or ζ-regularised) trace of B obtained as the finite part at $z = 0$ of the meromorphic expansion $\mathrm{TR}(B(D^2 + \pi_D)^{-z})$ where B is a form valued pseudodifferential operator and TR the canonical trace on non-integer order pseudodifferential operators [10]. π_D stands for the orthogonal projection onto the kernel of D.

This "renormalisation" procedure applies to the geometric set-up corresponding to families of Dirac operators associated with a trivial fibration of manifolds (see Theorem 1).

In the case of a family of Dirac operators associated with a general fibration of manifolds, such a straightforward "renormalisation procedure" is not possible due to the presence of an extra curvature term arising from a horizontal distribution on the fibration. Indeed, the Chern–Weil forms associated with a superconnection then differ from a weighted Chern form by (a priori) non-local terms involving this extra curvature term.

For a family of Dirac operators associated with a general fibration of spin manifolds $\pi : \mathbb{M} \to B$, on the grounds of the family index theorem, we identify Chern forms associated with the superconnection with form components of $\int_{\mathbb{M}/B} \hat{A}(\mathbb{M}/B) \wedge \mathrm{ch}(\mathbb{E}_{\mathbb{M}/B})$ where $\mathbb{E} \to \mathbb{M}$ is a vector bundle over $\mathbb{M}$. The j-th Chern form associated with a superconnection $\mathbb{A}$ introduced in [19] (following ideas of [22]) has $2j$-form part (see Theorem 2)

$$\mathrm{str}^{\mathbb{A}^2}(\mathbb{A}^{2j})_{[2j]} = \frac{(-1)^j j!}{(2i\pi)^{\frac{n}{2}}} \left(\int_{\mathbb{M}/B} \hat{A}(\mathbb{M}/B) \wedge \mathrm{ch}(\mathbb{E}_{\mathbb{M}/B}) \right)_{[2j]}.$$

On the grounds of the previous discussion, when the fibration is trivial, it differs from renormalised weighted Chern forms by local terms. As could be expected in analogy with the finite dimensional situation, in the graded case, the

[1] The only two up to linear combinations [11].