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Stretching in a model of a turbulent flow

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ABSTRACT

Using a multi-scaled, chaotic flow known as the KS model of turbulence [J.C.H. Fung, J.C.R. Hunt, A. Malik, R.J. Perkins, Kinematic simulation of homogeneous turbulence by unsteady random fourier modes, J. Fluid Mech. 236 (1992) 281–318], we investigate the dependence of Lyapunov exponents on various characteristics of the flow. We show that the KS model yields a power law relation between the Reynolds number and the maximum Lyapunov exponent, which is similar to that for a turbulent flow with the same energy spectrum. Our results show that the Lyapunov exponents are sensitive to the advection of small eddies by large eddies, which can be explained by considering the Lagrangian correlation time of the smallest scales. We also relate the number of stagnation points within a flow to the maximum Lyapunov exponent, and suggest a linear dependence between the two characteristics.

1. Introduction

Measures of stretching, such as the Lyapunov exponent, are important tools for understanding the nature of dynamical systems. For example, the maximum Lyapunov exponent can provide information about the complexity of an attractor via the Kaplan–Yorke dimension [6], or the rate of loss of information in the system [16]. The use of Lyapunov exponents in turbulent flows is far too great to list here; examples range from probing the onset of turbulence [3], to detecting inhomogeneity in hydromagnetic convection [9]. Another interesting application arises in dynamo theory; it was shown [15] that Lyapunov exponents provide an upper bound for the growth rate of a fast dynamo, and also a non-trivial combination of Lyapunov exponents gives an exact growth rate for the small scale turbulent dynamo [5].

In this work we use a model turbulent flow, known as the Kinematic Simulation (KS) model, that has been primarily used as a Lagrangian model of turbulence [7,8]. An important feature of KS is that it allows full control of the energy spectrum; moreover, its simple analytic structure means that numerical differentiation is not required in calculating the Lyapunov exponents. The KS model has been shown to be in good agreement with results obtained from direct numerical simulations (DNS) of turbulent flows, particularly with respect to Lagrangian statistics such as two-particle dispersion [8,10,11]. The use of the model is spreading rapidly to many other areas such as aeroacoustics and biomechanics. This flow has also been shown to be a hydromagnetic dynamo [17]. Motivated by the success of the KS model and its applications in magnetohydrodynamics, our aim is to check the agreement between the model and turbulent flows with respect to Lyapunov exponents. It has been shown that the KS model exhibits Lagrangian chaos [8,10]; we shall quantify this feature using the largest Lyapunov exponent.

2. The velocity field

The KS model prescribes the flow velocity at a position $\mathbf{x}$ and time t through the summation of Fourier modes with randomly chosen parameters. These modes are mutually independent, therefore the advection of small eddies by large eddies is not included in the model. More precisely, the velocity field is prescribed to be [11]

$$\mathbf{u}(\mathbf{x},t)=\sum_{n=1}^{N}\left(\mathbf{A}_n\times\mathbf{k}_n\cos\psi_n+\mathbf{B}_n\times\mathbf{k}_n\sin\psi_n\right), \qquad (1)$$

where $\psi_n = \mathbf{k}_n\cdot\mathbf{x}+\omega_n t$ and N is the number of modes. The unit vectors $\hat{\mathbf{k}}_n$ are chosen randomly, and $\mathbf{k}_n = k_n\hat{\mathbf{k}}_n$ where k_n is the wavenumber of the nth mode. The construction of $\mathbf{A}_n$ and $\mathbf{B}_n$,

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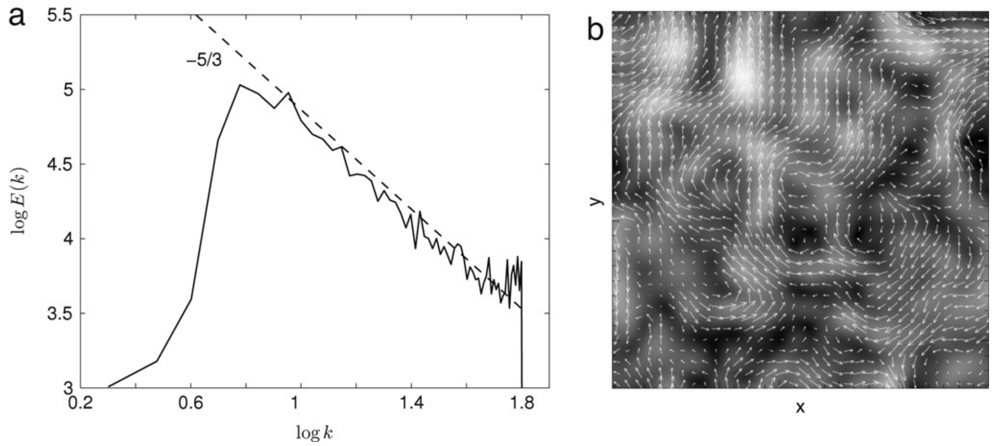

Fig. 1. **(a)** The energy spectrum, $E(k)$, showing the imposed $p = 5/3$ slope, as obtained by Fourier transform of Eq. (1) with $N = 20$, $k_1 = 10$ and $k_N = 400$. **(b)** Slice in the z plane of the vorticity field generated by taking curl of the velocity field from **(a)**, lighter shading indicates higher vorticity. Velocity vectors are shown in white.

which are time independent, is explained in the Appendix. Even though the parameters of the flow are chosen randomly, they do not necessarily change with time, so the flow is not necessarily random. We adopt a normalised energy spectrum of the KS flow $E(k)$, which is a modification of the von Kármán energy spectrum,

$$E(k) = k^4(1 + k^2)^{-(2+p/2)} e^{-1/2(k/k_N)^2}, \tag{2}$$

which reduces to $E(k) \propto k^{-p}$ in the inertial range $1 \ll k \ll k_N$, with $k = 1$ at the integral scale; $p = 5/3$ produces the Kolmogorov spectrum. As mentioned previously, a useful feature of the KS model is the ability to vary the slope p in the inertial range. The flow is incompressible and time dependent; the frequency of the nth mode, ω_n is inversely proportional to its turnover time,

$$\omega_n = \sqrt{k_n^3 E(k_n)}. \tag{3}$$

It is convenient to write the unit vector $\hat{\mathbf{k}}_n$ as

$$\hat{\mathbf{k}}_n = \begin{pmatrix} \sqrt{1-\zeta_n^2}\cos\theta_n \\ \sqrt{1-\zeta_n^2}\sin\theta_n \\ \zeta_n \end{pmatrix}, \tag{4}$$

where, $\theta_n \in [0, 2\pi)$ and $\zeta_n \in [-1, 1]$, are uniformly distributed random numbers, to ensure that $\hat{\mathbf{k}}_n$ are isotropically distributed. With

$$k_n = k_1 \left(\frac{k_N}{k_1}\right)^{(n-1)/(N-1)}, \tag{5}$$

the effective Reynolds number is introduced using the requirement that the dissipation and eddy turnover times are equal to each other at $k = k_N$,

$$\mathrm{Re} = (k_N/k_1)^{(p+1)/2}. \tag{6}$$

Since the maximum value of $E(k)$ and the integral scale remain unchanged in the models discussed here, any variation in Re can be thought to be caused by changes in the fluid viscosity. Fig. 1 shows the energy spectrum of the KS flow, obtained numerically after fast Fourier transforming $\mathbf{u}$ calculated from Eq. (1) on a 128^3 mesh. We also show a slice, in the z plane, of the corresponding vorticity field, with velocity vectors.

3. The Lyapunov exponents

To obtain the spectrum of Lyapunov exponents, λ_i, we measure the average rates of exponential divergence of nearby fluid particle trajectories. If the system is chaotic, at least one Lyapunov exponent is positive. The procedure to calculate the Lyapunov exponents consists of monitoring the evolution of an infinitesimal fluid sphere moving with the flow. The sphere, deformed by the flow, rapidly becomes an ellipsoid. Then the Lyapunov exponents are defined as

$$\lambda_i = \lim_{t\to+\infty} \frac{1}{t} \log_2 \frac{p_i(t)}{p_i(0)}, \tag{7}$$

where $p_i(t)$ is the ellipsoid's ith principal axis, and $i = 1, 2, 3$. Another feature of the KS flow is that it is time reversible (unlike 'real' turbulence), therefore the second Lyapunov exponent vanishes [2,1]. We now consider two remaining exponents, which must have opposite signs, $\lambda_1 = -\lambda_3$, since the flow is incompressible ($\nabla \cdot \mathbf{u} = 0$), the sum of the Lyapunov exponents must be zero. Hence we only need to calculate one exponent, $\lambda = \max(\lambda_i)$. Following Wolf et al. [18], consider a sphere whose centre, at $\mathbf{x}_0$, moves along a trajectory defined by

$$\frac{d}{dt}\mathbf{x}_0(t) = \mathbf{u}(\mathbf{x}_0, t), \tag{8}$$

with $\mathbf{u}$ obtained from Eq. (1). As the sphere follows a trajectory in the flow, its shape is deformed to an ellipsoid by stretching and compression. To the linear approximation in the sphere radius η, $\mathbf{x}_0$ remains the centre of the deformed ellipsoid. The positions of the points on the surface of the sphere $\boldsymbol{\eta} = \mathbf{x} - \mathbf{x}_0$, where $\mathbf{x}_0$ is the position of the centre of the ellipsoid, obey the linearised equations of motion

$$\frac{d}{dt}\eta_i(t) = D_{ij}\eta_j, \tag{9}$$

where $D_{ij} = \partial u_i/\partial x_j$ and the summation convention is assumed. We integrate Eqs. (8) and (9) numerically, normalising $\boldsymbol{\eta}$ at regular intervals as to keep the linearisation valid. We then take the temporal average of the magnitude of $\boldsymbol{\eta}$ to recover the maximum Lyapunov exponent. Finally, we average the results over 500 particles to improve the statistics. Since detailed behavior of λ can vary significantly between different realisations of the flow, we further take an ensemble average over 50 different realisations of the KS model with the same non-random parameters. The results of one such run are shown in Fig. 2. Before beginning the simulations, the code was tested by computing Lyapunov exponents for some well known chaotic flows [13].