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On ergodic and mixing properties of the triangle map

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ABSTRACT

In this paper, we study in detail, both analytically and numerically, the dynamical properties of the triangle map, a piecewise parabolic automorphism of the two-dimensional torus, for different values of the two independent parameters defining the map. The dynamics is studied numerically by means of two different symbolic encoding schemes, both relying on the fact that it maps polygons to polygons: in the first scheme we consider dynamically generated partitions made out of suitable sets of disjoint polygons, in the second we consider the standard binary partition of the torus induced by the discontinuity set. These encoding schemes are studied in detail and shown to be compatible, although not equivalent. The ergodic properties of the triangle map are then investigated in terms of the Markov transition matrices associated to the above schemes and furthermore compared to the spectral properties of the Koopman operator in $L^2(\mathbb{T}^2)$. Finally, a stochastic version of the triangle map is introduced and studied. A simple heuristic analysis of the latter yields the correct statistical and scaling behaviours of the correlation functions of the original map.

1. Introduction

The research of conservative dynamical systems has a long and fruitful history. From the discovery of the exponential sensitivity on initial conditions by Poincaré the main part of research directed its attention to such *chaotic* situations. The Kolmogorov–Sinai entropy is one of the basic invariants of dynamical systems, which divides them into two classes: (i) the chaotic or hyperbolic systems, with positive dynamical entropy, and (ii) non-hyperbolic systems, with zero dynamical entropy. A particular subclass of non-hyperbolic systems has been discovered recently, exhibiting properties common mostly to chaotic systems such as diffusion, ergodicity and mixing. The study of such systems is important in order to understand the origin of stochastic-like-dynamics in non-hyperbolic dynamical systems – and in particular in non-equilibrium statistical mechanics – and the effects on their quantum counterparts. A physically relevant example in this class is the triangle map [1]. As we will see below, this is a two parameter (α, β) family of two dimensional maps on the torus. They are related to the returning map of a point particle moving inside an elongated rectangular triangle billiard, namely for the correspondence to be accurate one of the angles of the triangular billiard has to be small. Roughly speaking α is related to one angle in the triangle and β is introduced in order to generalize the map and to make the dynamics richer. For the detailed construction of the map and its relation to the triangular billiards we refer to original articles [2,1]. For a more complete description of the dynamics in polygonal billiards see e.g. [3,4]. Some known generic properties of dynamics in polygonally shaped billiards are reviewed in [5]. The triangle map $\phi : \mathbb{T}^2 \to \mathbb{T}^2$ is a one-to-one transformation (automorphism) of a two-dimensional torus $\mathbb{T}^2 = [0, 1)^2$ written in additive notation (mod 1) as

$$\phi(q, p) = (q + p + \alpha\theta(q) + \beta, p + \alpha\theta(q) + \beta), \tag{1}$$

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where

$$\theta(q) = \begin{cases} 1 & : \quad q \in \left[0, \frac{1}{2}\right) \\ -1 & : \quad \text{otherwise} \end{cases}$$

with two parameters $(\alpha, \beta) \in \mathbb{R}^2$. The Jacobian matrix of the map reads

$$J = \frac{\partial \phi(q, p)}{\partial(q, p)} = \begin{pmatrix} 1 - 2\alpha \left\{\delta(q) + \delta\left(q - \frac{1}{2}\right)\right\} & 1 \\ -2\alpha \left\{\delta(q) + \delta\left(q - \frac{1}{2}\right)\right\} & 1 \end{pmatrix}$$

so that $\det J = 1$ and $\operatorname{tr} J = 2[1 - \alpha \left\{\delta(q) + \delta\left(q - \frac{1}{2}\right)\right\}]$. The map is area-preserving and piecewise parabolic. However it is not continuous, the 'discontinuity set' $\mathcal{D}$ is the following codimension one manifold

$$\mathcal{D} = \{(0, p) \mid p \in [0, 1)\} \cup \left\{\left(\frac{1}{2}, p\right) \mid p \in [0, 1)\right\}$$
$$=: v(0) \cup v\left(\frac{1}{2}\right).$$

The action of the map ϕ can be decomposed into three simple transformations:

(1) Cut of the torus along two vertical lines $v(0)$ and $v(\frac{1}{2})$ and translation of the two resulting pieces in opposite directions parallel to the cuts: $q \to q$ and $p \to p + \alpha\theta(q)$;
(2) Rigid translation by β along the p-direction: $q \to q$ and $p \to p + \beta$;
(3) Parabolic skew translation: $q \to q + p$ and $p \to p$.

The triangle map defines a time-discrete dynamical system ϕ^t for $t \in \mathbb{Z}$ by a recursion relation

$$\mathbf{x}_t = \phi(\mathbf{x}_{t-1}) = \phi^t(\mathbf{x}_0), \quad \mathbf{x}_t = (q_t, p_t). \tag{2}$$

where the t-th iteration of the map can be explicitly written as

$$q_t(q_0, p_0) = q_0 + p_0 t + \frac{\beta}{2} t(t + 1) + \alpha \sum_{k=1}^{t-1} S_k,$$
$$p_t(q_0, p_0) = p_0 + \beta t + \alpha S_t, \tag{3}$$

where

$$S_t := \sum_{k=0}^{t-1} \theta(q_k) \quad \text{satisfies } S_{t+1} = S_t \pm 1. \tag{4}$$

Note that $S_t \in \mathbb{Z}$ and $|S_t| \le t$.

Using (3) and the fact that the function $\theta(q)$ is locally constant, it is easy to see that the dynamics transforms a horizontal segment to a finite number of horizontal segments. On the other hand, for each t, the image of a given vertical line $v(q_0) = \{(q_0, p), p \in (0, 1]\}$ under the map ϕ^t is a family of segments having a slope $1/t$ and both the length and the position are determined by the pseudo-random series $\sum_{k=1}^{t-1} k\,(\beta + \alpha\,\theta(q_k))$.

Away from the singular set $\mathcal{D}$, the map ϕ acts locally as a linear stretching: a small parallelogram P, which is bounded by two horizontal sides and two non-horizontal sides forming an angle γ with the horizontal axis, is mapped to another parallelogram P'. P' has the same area as P and is again bounded by two horizontal sides of unchanged length, whereas the other two sides are rotated clockwise forming an angle $\gamma' = \gamma/(1 + \gamma)$ with the horizontal axis, so that they get both stretched by a factor $\sin\gamma \cdot \sqrt{1 + (1 + \cot\gamma)^2}$.

Two nearby points (q, p) and $(q + \Delta q, p + \Delta p)$ on the same side of discontinuity are evolving so that

$$\Delta(t) = \|\phi^t(q + \Delta q, p + \Delta p) - \phi^t(q, p)\|$$
$$= \sqrt{\Delta p^2 + (\Delta q + t\,\Delta p)^2}. \tag{5}$$

Notice that the distance between points lying on a horizontal segment ($\Delta p = 0$) does not grow with time, $\Delta(t) = |\Delta q|$, whereas for points on a vertical segment ($\Delta q = 0$) the distance grows as $\Delta(t) = |\Delta p|\sqrt{1 + t^2}$. This implies that two arbitrary points on the same horizontal segment can be separated only by the cutting mechanism on the singular line. This implies that the points in a horizontal line are not separated by stretching, but rather by chance that they fall with time on opposite sides of discontinuity, which is guaranteed by the stochastic like behaviour which shall be discussed in the sequel.

The triangle map has been further explored and applied in investigations of fundamental properties of statistical mechanics in several recent papers [6–8]. In the following text we study, mainly numerically, ergodic properties of the triangle map using two different ways to symbolically encode the dynamics, see e.g [9], called the polygonal and the binary description. We establish some common properties of the two descriptions as well as some their specific features. In the frame of a given description we study a finite state Markov chain corresponding to the triangle map and its spectral gap. We also analyze certain interesting scaling relations of the spectrum of the Koopman operator in the truncated Fourier basis. Additionally we draw parallels between the triangle map and its stochastic version called random triangle map. The later possesses similar properties as the deterministic version of the triangle map, but enables analytical predictions thanks to the possibility of averaging over different realizations.

2. Some further properties of the triangle map

The ergodic properties of the triangle map system strongly depend on the arithmetic properties of the parameters α and β. In the following we list all the analyzed cases and present the results.

2.1. The case $\alpha = 0$

If $\alpha = 0$ the map reduces to the skew translation $\phi(q, p) = (q + p + \beta, p + \beta)$. As is well known, for β irrational the map is (uniquely) ergodic but not even weakly mixing (see reference e.g. [10] and below)

2.2. The case α and β rational

If α and β are rational numbers then the dynamics is *pseudo-integrable* [11]. For example, if $\alpha = n/m$, and $\beta = r/m$, for some integers n, m, r such that $\gcd(n, m, r) = 1$, then the phase space $\mathbb{T}^2$ is foliated into *invariant curves* $\{(q, p + k/m) | q \in [0, 1), k \in \{0, 1, \ldots, m - 1\}\}$. It is easy to see that similar foliations exist also for arbitrary rational values of the parameters and the dynamics ϕ restricted on each set of invariant curves is just an *interval exchange transformation* (IET). See Lemma 4 for a more precise statement.

2.3. The generic case

The case in which both parameters are non zero and irrational, i.e. $0 \ne \alpha, \beta \in \mathbb{R} \setminus \mathbb{Q}$, with $\beta \notin \mathbb{Z} + \alpha\mathbb{Z}$, will be referred to as the *generic case*. We have the following

Lemma 1. *In the generic case ϕ has no periodic orbits.*

Proof. According to (3), a necessary condition for ϕ to have a periodic point (q_0, p_0) of period $t \ge 1$ is that $\beta t + \alpha S_t = 0 \pmod 1$. But this is clearly impossible under the assumption of the lemma. $\square$