Available online at www.sciencedirect.com
SCIENCE DIRECT®
ELSEVIER
Journal of Geometry and Physics 56 (2006) 1096–1115
JOURNAL OF GEOMETRY AND PHYSICS
www.elsevier.com/locate/jgp

Mirror symmetry and generalized complex manifolds
Part II. Integrability and the transform for torus bundles

Oren Ben-Bassat*

Department of Mathematics, University of Pennsylvania, Philadelphia, PA 19104-6395, USA

Received 31 March 2005; accepted 11 April 2005
Available online 1 August 2005

Abstract

In this paper we continue the development of a relative version of T-duality in generalized complex geometry which we propose as a manifestation of mirror symmetry. We discuss the integrability of the transform from Part I in terms of data on the base manifold. We work with semi-flat generalized complex structures on real n-torus bundles with section over an n-dimensional base and use the transform on vector bundles developed in Part I of this paper to discuss the bijective correspondence between semi-flat generalized complex structures on pairs of dual torus bundles. We give interpretations of these results in terms of relationships between the cohomology of torus bundles and their duals. We comment on the ways in which our results generalize some well established aspects of mirror symmetry. Along the way, we give methods of constructing generalized complex structures on the total spaces of the bundles we consider.

MSC: 53Z05; 14D06; 53D05

Keywords: Mirror symmetry; T-duality; 14J32; Generalized complex manifolds; Dirac manifolds

* Tel.: +1 215 898 5980.
E-mail address: orenb@math.upenn.edu.

0393-0440/$ – see front matter
doi:10.1016/j.geomphys.2005.04.020

1. Introduction

In the associated paper, Part I: The Transform on Vector Bundles, Spinors, and Branes [1], we gave transformation rules for generalized almost complex structures on vector bundles, including various assumptions and compatibility conditions. We have also commented on the mirror transformation on spinors and branes, as well as a relationship to Dirac geometries on the base manifold. Furthermore, we have examined the geometry of a pair of transverse foliations, and the compatibilities with generalized complex and generalized Kähler structures. We have proposed this transformation as a very simple case of mirror symmetry. In this paper, we continue the analysis, focusing on the integrability conditions, and the new features that arise in the case of torus bundles. Finally, we work out some more explicit details in certain examples. In Part I, we have included a more complete introduction, with background material and additional references.

In this paper, we relate the integrability of *semi-flat* (see Definition 2.2) generalized almost complex structures on torus and vector bundles to data which lives only on the base manifold. We show that a semi-flat generalized almost complex structure is integrable if and only its mirror structure is integrable.

Using a natural connection on a torus bundle $Z \to M$ with zero section s, we will construct semi-flat generalized complex structures $\mathcal{J}$ on Z from generalized almost complex structures $\underline{\mathcal{J}}$ on the vector bundle $s^*T_{Z/M} \oplus T_M$. The definition of semi-flat includes the condition that

$$\underline{\mathcal{J}}(s^*T_{Z/M} \oplus s^*T^\vee_{Z/M}) = T_M \oplus T^\vee_M.$$

Then we have the following two results.

Theorem 1.1 (3.4). *A semi-flat generalized almost complex structure $\mathcal{J}$ on a torus bundle $Z \to M$ with zero section s is integrable if and only if*

$$[\underline{\mathcal{J}}(\mathcal{S} \oplus \mathcal{S}^\vee), \underline{\mathcal{J}}(\mathcal{S} \oplus \mathcal{S}^\vee)] = 0,$$

where $\mathcal{S}$ is the sheaf of flat sections of $s^*T_{Z/M}$.

Corollary 1.2 (3.5). *A semi-flat generalized almost complex structure $\mathcal{J}$ on a torus bundle $Z \to M$ is integrable if and only if its mirror structure $\hat{\mathcal{J}}$ on the dual torus bundle $\hat{Z} \to M$ is integrable.*

These statements set the stage for understanding mirror symmetry and the mirror transform of D-branes in generalized Calabi–Yau geometry. Our results are a direct generalization of the setup employed by Arinkin and Polishchuk [32] in ordinary mirror symmetry. Explicit examples of this fact can be found in Section 5.

We relate this transformation of geometric structures to a purely topological map on differential forms which descends to a map from the de Rham cohomology of Z to the de Rham cohomology of $\hat{Z}$. In particular, the map on differential forms exchanges the pure spinors associated to the generalized complex structure on Z with the ones associated to the