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On spacelike hypersurfaces of constant sectional curvature lorentz manifolds

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Abstract

Let $x : M^n \to \bar{M}^{n+1}$ be an n-dimensional spacelike hypersurface of a constant sectional curvature Lorentz manifold $\bar{M}$. Based on previous work of S. Montiel, L. Alías, A. Brasil and G. Colares studied what can be said about the geometry of M when $\bar{M}$ is a conformally stationary spacetime, with timelike conformal vector field K. For example, if M^n has constant higher order mean curvatures H_r and H_{r+1}, they concluded that M^n is totally umbilical, provided $H_{r+1} \neq 0$ on it. If $\mathrm{div}(K)$ does not vanish on M^n they also proved that M^n is totally umbilical, provided it has, a priori, just one constant higher order mean curvature.

In this paper, we compute $L_r(S_r)$ for such an immersion, and use the resulting formula to study both r-maximal spacelike hypersurfaces of $\bar{M}$, as well as, in the presence of a constant higher order mean curvature, constraints on the sectional curvature of M that also suffice to guarantee the umbilicity of M. Here, by L_r we mean the linearization of the second order differential operator associated to the r-th elementary symmetric function S_r on the eigenvalues of the second fundamental form of x.

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1. Introduction

In the past 30 years, there has been an increasing interest in studying the structure of spacelike hypersurfaces of Lorentz manifolds of constant sectional curvature. This goes back to 1976, when S.Y. Cheng and S.T. Yau proved ([8]) the Calabi–Bernstein conjecture concerning complete maximal spacelike hypersurfaces of the Lorentz–Minkowsky space, namely, that the only ones are the spacelike hyperplanes.

For the De Sitter space, A.J. Goddard conjectured in [11] that complete spacelike hypersurfaces having constant mean curvature should be totally umbilical. Although the original problem turned out to be false in general, the efforts to prove it motivated a great deal of work by several authors, trying to figure out what additional geometric restrictions should be imposed in the hypersurface to get an affirmative answer. Goddard's conjecture was eventually proved to be true for the case of closed hypersurfaces, due to independent work of S. Montiel ([16]) and J.L. M. Barbosa and V. Oliker ([6]).

In recent years, the main stream of investigation has turned towards more general classes of Lorentz ambient spaces, dealing mostly with the problems of existence and uniqueness of constant mean curvature spacelike hypersurfaces. In [4], the authors proved that the only closed spacelike hypersurfaces of generalized Robertson–Walker spacetimes satisfying a suitable condition are the totally umbilical ones. By such spaces we mean warped products $I \times_f F^n$, where $I \subset \mathbb{R}$ is an open interval with the metric $-\mathrm{d}t^2$, F^n is an n-dimensional Riemannian manifold and $f : I \to \mathbb{R}$ is a positive smooth function. Note the these include both the Lorentz–Minkowsky space and the De Sitter space. Later on, S. Montiel considered (in [17]) the same problem for conformally stationary spacetimes, that is, Lorentz manifolds possessing a closed conformal timelike vector field K, where by closed we mean that the dual one form ω^K of K is closed. This class of spaces includes the previous one, for $K = f\frac{\partial}{\partial t}$ is a closed conformal timelike vector field in $I \times_f F^n$.

Lately, in [3], the authors studied what can be said about the geometry of a closed spacelike hypersurface M^n of a conformally stationary spacetime $\overline{M}^{n+1}$ if one imposes constraints on higher order mean curvatures of M. Among other results, they proved that if M is contained in a region of $\overline{M}$ where the divergence of the timelike conformal vector field K does not vanish, then M is totally umbilical provided it has, a priori, just one constant higher order mean curvature. In the De Sitter space, for example, this amounts for M being contained in the future or chronological past of an equator, thus agreeing with previous results in the literature. They also proved that M is totally umbilical provided it has two consecutive constant higher order mean curvatures H_r and H_{r+1}, with $H_{r+1} \neq 0$ on it (actually, this hypothesis is missing there).

Their method, which consists in applying certain integral formulae involving the higher order mean curvatures of M together with the classical Newton's inequalities (see [13]), has the disadvantage of not working for complete hypersurfaces. Moreover, in either the complete or compact case, asking what could be said of M once one has dropped the condition of the nonvanishing of the divergence of K is a question that naturally arises at this point. In particular, what can be said of r-maximal spacelike hypersurfaces of $\overline{M}$?

In this paper we give partial answers to these questions. Our approach is to compute $L_r(S_r)$ for a spacelike hypersurface $x : M^n \to \overline{M}^{n+1}$ of a time-oriented Lorentz manifold with no additional tructure, applying the resulting formulae in the study of the case of