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Quasi-periodic stability of normally resonant tori

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ABSTRACT

We study quasi-periodic tori under a normal-internal resonance, possibly with multiple eigenvalues. Two non-degeneracy conditions play a role. The first of these generalizes invertibility of the Floquet matrix and prevents drift of the lower dimensional torus. The second condition involves a Kolmogorov-like variation of the internal frequencies and simultaneously versality of the Floquet matrix unfolding. We focus on the reversible setting, but our results carry over to the Hamiltonian and dissipative contexts.

1. Introduction

Persistence results for quasi-periodic motions were first proved for maximal tori in Hamiltonian systems and became known as Kolmogorov–Arnol'd–Moser (KAM) theory. In [31] this was extended to lower dimensional tori and to other contexts like volume preserving and reversible systems. The rôle of the 'modifying terms' in terms of system parameters was clarified in [14,24] and the Rüssmann condition [13,35] allows one to subsequently reduce the high number of parameters to the bare minimum.

These results yield what is called quasi-periodic (or normal linear) stability, i.e. families of invariant tori persist under sufficiently small perturbations when restricted to certain (measure-theoretically large) Cantor sets. The theorems in [13] make the crucial assumption that all eigenvalues of the matrix Ω describing the normal linear behavior be simple. This implies in particular that $\det \Omega \neq 0$ (except for the dissipative case and the high-dimensional volume preserving case, where this condition is explicitly added). Multiple resonances are admitted in [11,17,22] and the aim of the present paper is to admit zero eigenvalues without weakening the conclusion of quasi-periodic stability.

1.1. Setting and results

We work on the phase space $M = \mathbb{T}^n \times \mathbb{R}^m \times \mathbb{R}^{2p}$, where $\mathbb{T}^n = (\mathbb{R}/2\pi\mathbb{Z})^n$ is the n-torus on which we use coordinates $x = (x_1, \ldots, x_n) (\mathrm{mod}\ 2\pi)$, while on $\mathbb{R}^m$ and $\mathbb{R}^{2p}$ we use respectively $y = (y_1, \ldots, y_m)$ and $z = (z_1, \ldots, z_{2p})$. In such coordinates a vector field on M takes the form

$$\dot{x} = f(x, y, z), \qquad \dot{y} = g(x, y, z), \qquad \dot{z} = h(x, y, z),$$

or in vector field notation:

$$X(x, y, z) = f(x, y, z)\partial_x + g(x, y, z)\partial_y + h(x, y, z)\partial_z. \tag{1.1}$$

We assume that the vector field X depends analytically on all variables, including possible parameters which we suppress for the moment; referring to [14,24,33] we note that our results remain valid when 'analyticity' is replaced by 'a sufficiently high degree of differentiability'. An invariant torus T of a vector field X is called parallel if a smooth conjugation exists of the restriction $X|_T$ with a constant vector field $\dot{x} = \omega$ on $\mathbb{T}^n$. The vector $\omega = (\omega_1, \omega_2, \ldots, x_n) \in \mathbb{R}^n$ is the (internal) frequency vector of T. The parallel torus is quasi-periodic when the frequencies are independent over the rationals.

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We are concerned with persistence of quasi-periodic tori under small perturbations, and to fix thoughts we concentrate[1] on the reversible setting. To define reversibility we consider an involution (i.e. $G^2 = I$)

$$G : M \longrightarrow M, \quad (x, y, z) \mapsto (-x, y, Rz), \tag{1.2}$$

with $R \in \mathrm{GL}(2p, \mathbb{R})$ a linear involution on $\mathbb{R}^{2p}$ such that

$$\dim \mathrm{Fix}(R) = \dim \left\{ z \in \mathbb{R}^{2p} \mid Rz = z \right\} = p.$$

The vector field X is then called G-reversible (or reversible for short) if

$$G_*(X) = -X.$$

Using (1.1) this reversibility condition takes the explicit form

$$f(-x, y, Rz) = f(x, y, z),$$
$$g(-x, y, Rz) = -g(x, y, z),$$
$$h(-x, y, Rz) = -Rh(x, y, z),$$

valid for all $(x, y, z) \in M$.

Following [12–14,24] the vector field X is called integrable if it is equivariant with respect to the group action

$$\mathbb{T}^n \times M \longrightarrow M, \quad (\xi, (x, y, z)) \mapsto (\xi + x, y, z)$$

of $\mathbb{T}^n$ on M, or in other words, if the functions f, g and h in (1.1) are independent of the x-variable(s). Such an integrable vector field

$$X(x, y, z) = f(y, z)\partial_x + g(y, z)\partial_y + h(y, z)\partial_z \tag{1.3}$$

is reversible if

$$f(y, Rz) = f(y, z), \qquad g(y, Rz) = -g(y, z) \qquad \text{and}$$
$$h(y, Rz) = -Rh(y, z) \tag{1.4}$$

for all $(y, z) \in \mathbb{R}^m \times \mathbb{R}^{2p}$; this implies $g(y, z) = 0$ for all $(y, z) \in \mathbb{R}^m \times \mathrm{Fix}(R)$. In case $h(0, 0) = 0$ the[2] n-torus $T_0 = \mathbb{T}^n \times \{0\} \times \{0\}$ is invariant under the flow of the vector field X. The normal linear part $N(X)$ of (1.4) at T_0 is given by

$$N(X)(x, y, z) = \omega\partial_x + \Omega z \, \partial_z, \tag{1.5}$$

with

$$\omega = f(0, 0) \quad \text{and} \quad \Omega = D_z h(0, 0).$$

We denote the subspace of infinitesimally reversible linear operators on $\mathbb{R}^{2p}$ by $\mathfrak{gl}_-(2p; \mathbb{R})$ and by $\mathfrak{gl}_+(2p; \mathbb{R})$ the subspace of all R-equivariant linear operators on $\mathbb{R}^{2p}$, i.e.

$$\mathfrak{gl}_\pm(2p; \mathbb{R}) = \{\Omega \in \mathfrak{gl}(2p; \mathbb{R}) \mid \Omega R = \pm R\Omega\}.$$

In order to define the non-degeneracy of (1.3) at the invariant torus T_0 we consider the subspaces

$$\mathcal{X}^{\pm G}_{lin} = \left\{ \omega\partial_x + \Omega z\partial_z \mid \omega \in \mathbb{R}^n, \Omega \in \mathfrak{gl}_\pm(2p; \mathbb{R}) \right\}$$

of the spaces $\mathcal{X}^{-G}$ of all G-reversible vector fields on M and $\mathcal{X}^{+G}$ of all G-equivariant vector fields, satisfying $G_*(X) = +X$. For $X \in \mathcal{X}^{-G}$ the adjoint operator

$$\mathrm{ad}\, N(X) : \mathcal{X} \longrightarrow \mathcal{X}, \quad Y \mapsto [N(X), Y]$$

maps $\mathcal{X}^{\pm G}$ into $\mathcal{X}^{\mp G}$; a similar statement is true for $\mathcal{X}^{\pm G}_{lin}$.

Our interest concerns purely G-reversible vector fields, and G-reversible vector fields that are furthermore equivariant with respect to

$$F_l : M \longrightarrow M, \quad (x, y, z) \mapsto \left(x_1 - \frac{2\pi}{l}, x_*, y, z_I, e^{\frac{2\pi i}{l}} z_{II} \right). \tag{1.6}$$

Here $z_{II} \cong z_{2j-1} + iz_{2j}$ singles out two of the z-variables in a complexified form and $z_I = (z_1, z_2, \ldots, z_{2j-2}, z_{2j+1}, \ldots, z_{2p})$ contains the remaining z-variables. To allow for a unified formulation of our results we define a reversing symmetry group Σ and a character (a group homomorphism) $\chi : \Sigma \longrightarrow \{\pm 1\}$ as follows:

(i) In the purely reversible case we set $\Sigma := \{\mathrm{Id}, G\}$ and $\chi(G) := -1$.
(ii) In the equivariant-reversible case we define Σ as the group generated by G and F_l and define χ by $\chi(G) := -1$ and $\chi(F_l) := 1$.

In both cases Σ is isomorphic to $\mathbb{Z}_2 \ltimes Z_l$, the dihedral group of order $2l$. When $l = 1$ the generator $F_1 = \mathrm{Id}$ of course is superfluous. For both cases we put

$$\mathcal{X}^+ = \{X \in \mathcal{X} \mid E_*(X) = X \text{ for all } E \in \Sigma\}$$
$$\mathcal{X}^- = \{X \in \mathcal{X} \mid E_*(X) = \chi(E)X \text{ for all } E \in \Sigma\}$$

together with $\mathcal{X}^\pm_{lin} = \mathcal{X}^{\pm G}_{lin} \cap \mathcal{X}^\pm$. Furthermore we let $\mathcal{B}^+$ and $\mathcal{B}^-$ consist of the constant vector fields in $\mathcal{X}^+$ and $\mathcal{X}^-$, respectively and denote by

$$\mathcal{O}(\Omega_0) = \left\{ \mathrm{Ad}\,(A) \cdot \Omega_0 := A\Omega_0 A^{-1} \mid A \in \mathrm{GL}_+(2p; \mathbb{R}) \right\}$$

the orbit under the adjoint action of $\mathrm{GL}_+(2p; \mathbb{R})$ on $\mathfrak{gl}_-(2p; \mathbb{R})$.

Definition 1 (*Broer, Huitema and Takens [14]*). The parametrized[3] vector field X_λ with linearization $N(X_\lambda)(x, y, z) = \omega(\lambda)\partial_x + \Omega(\lambda)z\partial_z$ is *non-degenerate* at $\lambda = \lambda_0 \in \mathbb{R}^s$ if

BHT(i) $\ker \mathrm{ad}\, N(X_{\lambda_0}) \cap \mathcal{B}^+ = \{0\}$;
BHT(ii) at $\lambda = \lambda_0$ the mapping $(\omega, \Omega) : \mathbb{R}^s \longrightarrow \mathbb{R}^n \times \mathfrak{gl}_-(2p; \mathbb{R})$, $\lambda \mapsto (\omega(\lambda), \Omega(\lambda))$ is transverse to $\{\omega(\lambda_0)\} \times \mathcal{O}(\Omega(\lambda_0))$.

The two non-degeneracy conditions BHT(i) and BHT(ii) generalize the condition that $\mathrm{ad}\, N(X_{\lambda_0})$ has to be invertible, a requirement that lies at the basis of Mel'nikov's conditions ((1.7) with $|\ell| \neq 0$). One also speaks of BHT non-degeneracy. Compared to the formulation in [14], Section 8a2 the requirement that $\Omega(\lambda_0)$ have only simple eigenvalues is dropped. The extension to multiple normal frequencies was developed in [11,17,22] for invertible $\Omega(\lambda_0)$; we return to the original formulation of BHT(i).

To formulate the strong non-resonance condition necessary for persistence of invariant tori we introduce for $\Omega \in \mathfrak{gl}_-(2p; \mathbb{R})$ the normal frequency mapping $\alpha : \mathfrak{gl}_-(2p; \mathbb{R}) \longrightarrow \mathbb{R}^{2p}$ where the components of $\alpha(\Omega)$ are equal to the imaginary parts of the eigenvalues of $\Omega \in \mathfrak{gl}_-(2p; \mathbb{R})$. Higher multiplicities are taken into account by repeating each eigenvalue as many times as necessary.

Definition 2. A pair $(\omega, \Omega) \in \mathbb{R}^n \times \mathfrak{gl}_-(2p; \mathbb{R})$ is said to satisfy a *Diophantine condition* if there exist constants $\tau > n - 1$ and $\gamma > 0$ such that

$$|\langle k, \omega \rangle + \langle \ell, \alpha(\Omega) \rangle| \geq \gamma |k|^{-\tau} \tag{1.7}$$

for all $k \in \mathbb{Z}^n \setminus \{0\}$ and $\ell \in \mathbb{Z}^{2p}$ with $|\ell| \leq 2$.

[1] We give explicit formulations for reversible vector fields, but the results remain valid for e.g. dissipative, Hamiltonian or volume-preserving systems (vector fields and maps), where equivariance is also optional.

[2] Often one has a whole family $T_y = \mathbb{T}^n \times \{y\} \times \{0\}$ of invariant tori. While we are especially interested in bifurcations, the variable y will still act as a parameter, now unfolding the bifurcation scenario.

[3] The rôle of the external parameter λ occurring in Definition 1 can be (partially) taken by the internal parameter y.