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Mixed-Mode Oscillations in a piecewise linear system with multiple time scale coupling

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HIGHLIGHTS

- Study of a four dimensional piecewise linear system with three different time scales.
- Proof of existence of an attractive limit cycle with complex Mixed-Mode oscillations.
- Interacting canard phenomena underlying changes and exchanges in oscillations number.

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ABSTRACT

In this work, we analyze a four dimensional slow–fast piecewise linear system with three time scales presenting Mixed-Mode Oscillations. The system possesses an attractive limit cycle along which oscillations of three different amplitudes and frequencies can appear, namely, small oscillations, pulses (medium amplitude) and one surge (largest amplitude). In addition to proving the existence and attractiveness of the limit cycle, we focus our attention on the canard phenomena underlying the changes in the number of small oscillations and pulses. We analyze locally the existence of secondary canards leading to the addition or subtraction of one small oscillation and describe how this change is globally compensated for or not with the addition or subtraction of one pulse.

1. Introduction

In [1], we considered a piecewise linear (PWL) version of the system initially proposed in [2,3], a model introduced in a context of mathematical neuroendocrinology. The original smooth model in [2,3] consists of a four dimensional (4D) system made of two coupled FitzHugh–Nagumo subsystems running on different timescales, in which the fast subsystem is forced by the slow subsystem. We replaced in [1] the original FitzHugh–Nagumo subsystems [4,5] by two PWL equivalents (McKean caricatures [6]). This change allowed us to obtain more information on the dynamics as well as a more direct control of the quantitative features of the system output. A typical orbit of the PWL model possesses a periodic behavior with oscillations of two different amplitudes and frequencies: a series of pulses (medium amplitude) and one surge (larger amplitude).

The smooth model, with three different time scales, can display Mixed-Mode Oscillations (MMOs) of three different types [7]. Apart from the pulses and surge, a pause may exist between the end of the surge and the resumption of the pulses, where small oscillations take place, see Fig. 1 in [7]. However, the McKean caricatures do not capture the existence of such small oscillations. Here, we propose an extension of the model considered in [1]. In the new extended model, small oscillations can happen, see Fig. 1. Such small oscillations are underlain by the existence of a canard explosion in the fast system. To obtain a proper canard explosion in a PWL system, one small extra linearity zone is necessary, as it has been pointed out in [8–10].

The present work focuses on the mechanisms underlying the creation–destruction of the resulting small oscillations and describes how the addition or loss of one small oscillation after the surge is possibly compensated for by the loss or addition of one pulse before the surge. In the smooth model, such a compensation mechanism has been explored numerically [7] and

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it turned out to be too complicated to be studied analytically. The PWL nature of the system proposed in this work allows us to better understand the dynamical mechanisms than in the case of the smooth model. More specifically, we find two remarkable results. First, the number of small oscillations needed to be lost in order to win one pulse increases while we decrease the intermediate time scale of the system. Second, as the system goes through the canard explosion, one pulse disappears at the end of the pulsatility phase and another is created from a small oscillation at the beginning of the pulsatility phase. We describe in detail these facts along the paper and more specifically in Proposition 3 and Theorem 3. Moreover, fixing the time scales, we are able to control the number of small oscillations as well as the addition or subtraction of pulses just by tuning a regular parameter of the system.

We note that our method of counting small oscillations relies heavily on the presence of an attracting limit cycle which includes the sequence of small oscillations as a part of its trajectory. A more general method for counting small oscillations and computing secondary canards is provided in [10]. The main novelty of our paper is the study of the interaction of two canard phenomena, both occurring dynamically, one at the corner point of the fast nullcline and another at the small zone replacing the other critical point, see Fig. 2. The context of PWL systems, due to its simplicity and the availability of explicit formulas, provides a suitable context for this study.

Here, we consider the following class of 4D PWL slow–fast systems,

$$\begin{cases}\dot{x} = -y + f(x),\\ \dot{y} = \delta(x + a_2 + cX),\end{cases} \tag{1}$$

$$\begin{cases}\dot{X} = \delta(-Y + g(X)),\\ \dot{Y} = \varepsilon\delta(X + b_1 Y + b_2),\end{cases} \tag{2}$$

where

$$f(x) = \begin{cases}-x-2, & \text{if } x \le -1,\\ x, & \text{if } x \in (-1, 1-\sqrt{\delta}],\\ 1-\sqrt{\delta}, & \text{if } x \in (1-\sqrt{\delta}, 1+\sqrt{\delta}),\\ -x+2, & \text{if } x \ge 1+\sqrt{\delta},\end{cases} \tag{3}$$

and

$$g(X) = \begin{cases}-X-k_2-1, & \text{if } X \le -1,\\ k_2 X, & \text{if } |X| \le 1,\\ -X+k_2+1, & \text{if } X \ge 1,\end{cases}$$

with $(x, y, X, Y)^T \in \mathbb{R}^4$, $0 < \varepsilon, \delta \ll 1$, $b_1 \ge 0$, a_2, b_2, c, k_1, k_2 are real and positive, and the overdot denotes the derivative with respect to time variable t. Let us call the zones of subsystem (1) the left, center, extra or right zone, when the corresponding first variable x is, respectively, smaller that -1, between -1 and $1-\sqrt{\delta}$, between $1-\sqrt{\delta}$ and $1+\sqrt{\delta}$ or larger than $1+\sqrt{\delta}$ (see the PWL-nullcline in Fig. 2). Sometimes we will use the superscripts L, C, E and R, to specify the referred zone. Note that the size of the extra zone ($2\sqrt{\delta}$) has been chosen following [8] in order to give rise to a canard explosion in subsystem (1). In the same way, we call the zones of subsystem (2) the left, center, or right zone, when the corresponding first variable X is smaller than -1, between -1 and 1 or larger than 1. We also use the superscripts L, C and R, to specify the referred zone. Note that, for the sake of simplicity, we study here the case where the y-nullcline is vertical.

System (1)–(2) consists of two coupled planar oscillators, one in variables x, y and the other in variables X, Y. The coupling is one-way, that is, the subsystem in variables X, Y evolves independently and we call it the *forcing system*. Variable X is forcing the (x, y) 2D system by changing dynamically the location of the y-nullcline.

System (1)–(2) is slow–fast with three time scales, namely, 1, δ and $\varepsilon\delta$. The forcing system (X, Y) evolves more slowly than the *forced system* (x, y). Bearing this in mind, we call the forcing system the *slow system* and the forced system the *fast system*. Moreover, each of the two planar systems is also slow–fast, with slow variable y and Y, respectively, and fast variable x and X, respectively.

Let us first consider some assumptions, in order to obtain the MMOs which were observed in the smooth system in [7], and explain briefly the dynamical behavior of the system.

(H1) The forcing system (2) has a relaxation limit cycle. We can divide the limit cycle into four parts, (see Fig. 3):

I. $X \in (X_{\min}, -1),\ Y < g(X)$,
II. $X \in (-1, X_{\max}),\ Y < g(X)$,
III. $X \in (1, X_{\max}),\ Y > g(X)$,
IV. $X \in (X_{\min}, 1),\ Y > g(X)$.

Note that, as $\varepsilon \to 0$ the cycle is approaching the X-nullcline in the right and left zones and jumps almost horizontally when it reaches the middle zone. We use the approximation

$$X_{\min} \simeq -2k_2 - 1 - \varepsilon(1 + b_2 + b_1 k_2) \quad \text{and} \\ X_{\max} \simeq 2k_2 + 1 + \varepsilon(1 - b_2 + b_1 k_2), \tag{4}$$

valid for ε small enough.

(H2) $|a_2 - 1| < c$.

Under hypothesis (H2), the vertical nullcline crosses the separation line $x = -1$ rightwards at a time when $|X| < 1$, therefore, when X (considered as the bifurcation parameter) travels along the limit cycle with the fast motion, see Figs. 3 and 9. At the crossing time, a left canard superexplosion bifurcation occurs, that is, the system presents a bounded continuum of canard homoclinic orbits from the equilibrium point and by moving X in such a way that the equilibrium enters the central zone, one large relaxation oscillation limit cycle is created (see Figs. 4 and 5 in [11]). The term superexplosion comes from the fact that instead of successive canard cycles curves ending up in the relaxation cycle, in this case, there is a continuum of canard homoclinic cycles that take place all at the same time, and the relaxation cycle is born instantaneously from this continuum. For more details about the canard superexplosion bifurcation see [11].

Here, in contrast to [1], we do not impose any upper bound for c. As a consequence, the rightmost location of the y-nullcline can reach the right zone, hence a right canard explosion bifurcation (actually, the PWL equivalent) [8] can occur while X is along the relaxation limit cycle, see Fig. 9. The right canard explosion bifurcation takes place when the equilibrium point is located in the extra zone, that is, $x \in (1 - \sqrt{\delta}, 1 + \sqrt{\delta})$. We explain this case in detail in the following lines.

Remark 1. Note that, as it was explained in detail in Remark 4 of [1], tuning parameter b_1 enables us to adjust the pulse frequency, more specifically, the duration of the low-frequency part of the pulsatile regime with respect to that of the high frequency part. As in this article we are no more interested in fitting our model to biological specifications, we have fixed $b_1 = 0$ along all the simulations. Regarding parameter a_2, we only need to be sure that hypothesis (H2) $|a_2 - 1| < c$ is satisfied. We have chosen $a_2 = 2$ because in this case (H2) reads $c > 1$, but another value could be equally chosen.

The value of X determines the location of the y-nullcline, hence the location of the unique equilibrium point of the fast system (see Fig. 8, for instance). If the value of X is such that the equilibrium point is located in zone R or L, then the fast system has a stable node and the orbits are approaching it. If the location of X is such that the equilibrium point is located in the central zone, then the orbits oscillate across the four linearity zones around the equilibrium point. Finally, if the location of X is such that the equilibrium point