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Combustion waves in hydraulically resistant porous media in a special parameter regime

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HIGHLIGHTS

- We study the stability of fronts in a model of combustion in porous media.
- We assume that the Lewis number is chosen in a specific way.
- We use a combination of energy estimates and Evans function computations.
- We prove nonlinear stability under the condition that there is no unstable spectrum.

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ABSTRACT

In this paper we study the stability of fronts in a reduction of a well-known PDE system that is used to model the combustion in hydraulically resistant porous media. More precisely, we consider the original PDE system under the assumption that one of the parameters of the model, the Lewis number, is chosen in a specific way and with initial conditions of a specific form. For a class of initial conditions, then the number of unknown functions is reduced from three to two. For the reduced system, the existence of combustion fronts follows from the existence results for the original system. The stability of these fronts is studied here by a combination of energy estimates and numerical Evans function computations and nonlinear analysis when applicable. We then lift the restriction on the initial conditions and show that the stability results obtained for the reduced system extend to the fronts in the full system considered for that specific value of the Lewis number. The fronts that we investigate are proved to be either absolutely unstable or convectively unstable on the nonlinear level.

1. Introduction

Sivashinsky and his collaborators in [1] proposed a model describing combustion in inert porous media under condition of high hydraulic resistance,

$$\begin{aligned} T_t - (1-\gamma^{-1})P_t &= \epsilon T_{xx} + YF(T), \\ P_t - T_t &= P_{xx}, \\ Y_t &= \epsilon \mathrm{Le}^{-1} Y_{xx} - \gamma YF(T), \end{aligned} \tag{1}$$

where Y is the scaled concentration of the reactant in the reaction zone, P is the pressure, and T is the temperature. The specific heat ratio $\gamma > 1$, the Lewis number $\mathrm{Le} > 0$, and the ratio of pressure and molecular diffusivities $\epsilon > 0$ are physical characteristics of the fuel. The reaction rate $YF(T)$ may or may not have an ignition cut-off, that is, $F(T) = 0$ on an interval $[0, T_{ign}]$ and $F(T) > 0$ and increasing for $T > T_{ign}$. For $F(T)$ Lipschitz continuity is assumed everywhere, except at the ignition temperature $T = T_{ign}$. Papers [1–4] contain detailed explanations and the deduction of this system.

In [5] it is suggested to consider this system with initial conditions

$$T(0,x) = T_0(x), \qquad Y(0,x) = 1, \qquad P(0,x) = 0.$$

It is also assumed that ϵ is significantly smaller than other parameters, therefore a simplification of (1) is offered in the literature which is obtained by setting $\epsilon = 0$,

$$\begin{aligned} T_t - (1-\gamma^{-1})P_t &= YF(T), \\ P_t - T_t &= P_{xx}, \\ Y_t &= -\gamma YF(T). \end{aligned} \tag{2}$$

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We point out that the terms ϵT_{xx} and $\epsilon \mathrm{Le}^{-1} Y_{xx}$ in (1) are singular perturbations, so results obtained for the system (2) should be dealt with caution, as singularly perturbed systems in general may support a behavior significantly different than one observed in the limiting system.

This paper is devoted to the study of the stability of traveling fronts in the system (1). Traveling fronts are solutions of the underlying PDE (1) that have a form $T(x,t) = T(\xi), P(x,t) = P(\xi)$, $Y(x,t) = Y(\xi)$, with $\xi = x - ct$, where c is the a priori unknown front speed, and that asymptotically connect distinct equilibria at $\pm\infty$. These solutions are sought as solutions of the traveling wave ODEs

$$\begin{aligned} -cT' + c(1-\gamma^{-1})P' &= \epsilon T'' + YF(T), \\ P'' &= c(T' - P'), \\ cY' + \epsilon Y'' &= \gamma Y F(T), \end{aligned} \tag{3}$$

that satisfy boundary-like conditions at $\pm\infty$ which we describe below.

Generally speaking, the equilibria of the system (2) are states where $YF(T) = 0$, so they can be described as the states where there is no fuel $Y = 0$, or as the cold states $T < T_{ign}$. From physical considerations two of the equilibria are of interest, the completely burnt state $P = 1, T = 1, Y = 0$, and the unburnt state where all of the fuel is present $T = 0, P = 0, Y = 1$. So we consider the system (2) together with the boundary conditions

$$\begin{aligned} &P(-\infty) = 1, \quad T(-\infty) = 1, \quad Y(-\infty) = 0, \\ &T(+\infty) = 0, \quad P(+\infty) = 0, \quad Y(+\infty) = 1. \end{aligned}$$

The existence and uniqueness of fronts in the system (2) has already been established in [6], under the assumption on the parameters $0 < T_{ign} < 1 - \gamma^{-1}$. Moreover, it is already known [7], that the front in (2) that satisfies the boundary conditions above is unique, up to translation. In [2] it is shown that as ϵ approaches 0, the ϵ-dependent fronts in (2) converge to the fronts of

$$\begin{aligned} -cT' + c(1-\gamma^{-1})P' &= YF(T), \\ P'' &= c(T' - P'), \\ cY' &= \gamma Y F(T). \end{aligned} \tag{4}$$

It is also known [8] that the solution in the limiting system (3) persists as a unique solution of speed of order O(1) of the system (2) with $0 < \epsilon \ll 1$. We note that in [8] it is assumed that Le = 1, but this assumption can be removed because it does not affect the result of the paper in any way.

The stability of fronts in the system (2) has been addressed in [9]. There an Evans function approach was used to find parameter regimes where the front is absolutely unstable. In other words, it was shown that there are parameter regimes where small perturbations to the front grow exponentially fast, in the co-moving frame. More importantly, parameter regimes were found where the front is convectively unstable, which means that small perturbations to the front that are initially localized near the rest state $(P, T, Y) = (1, 1, 0)$ stay near that equilibrium.

To our knowledge, for no parameter values the stability of the traveling fronts in (1) (with $\epsilon > 0$) has been yet addressed. We point out that since the perturbation with small ϵ is singular, the stability (instability) of a front in the limiting system (2) does not directly imply the stability (instability) of the front in (1), even when ϵ is very small. While we follow the same standard sequence of steps as we did in [9] for the case $\epsilon = 0$, from a technical point of view our analysis is significantly different. Indeed, the case $\epsilon = 0$ is a singular limit of Model (1) in the sense that the order of the system is reduced by two (in [9], the order is furthermore reduced by one by using special initial conditions). It is well known that properties of existence and stability in a singular limit do not have to hold in general, even when ϵ is small (see for example [10]). In this paper, we consider another singular limit for Model (1). This limit is obtained by choosing a particular value for the Lewis number, in the presence of a strictly positive ϵ, and the initial conditions to satisfy (9). This choice reduces the order of Model (1) by two. One main difference with the limit $\epsilon = 0$ studied in [9] is that in the reduction of Model (1) that we consider here the dimension of the linearization is larger, making the Evans function computation more complicated. Namely, we had to use the definition of the Evans function that involves the wedge product as opposed to the definition corresponding to a linear system with a one-dimensional stable manifold as in [9], which is simpler and less numerically sensitive because the Evans function is defined by the scalar product of two solutions. Moreover, the nature of the energy estimates computed in both papers is completely different. This is because the orders of the systems studied in [9] and here are not the same.

As one would expect, the numerical calculation of the Evans function assumes a more precise definition of the reaction term than the one given at the beginning of the introduction, therefore we base our analysis on the assumption [11] of a discontinuous reaction rate is

$$F_d(T) = \begin{cases} \exp\left(Z\left\{\dfrac{T-h}{\sigma + (1-\sigma)T}\right\}\right), & T \geq T_{ign}, \\ 0, & T < T_{ign}, \end{cases} \tag{5}$$

where $Z > 0$ is the Zeldovich number, and $0 < \sigma < 1$ is the ratio of the characteristic temperatures of fresh and burned reactant.

The discontinuity in the reaction term is often introduced in combustion models [12] to account for the fact that for low temperatures the reaction rate is many orders less than the reaction rate at high temperatures.

However, to work with a well-defined linear operator obtained by linearizing the reaction term about the continuous front, we follow the recipe given in [9] and consider a smooth F, which is defined like F_d everywhere except for a small interval $(T_{ign}, T_{ign} + 2\delta)$ where the function is modified so as to go to zero in a smooth and monotonic fashion, for example, as

$$F_\delta(v) = \begin{cases} \exp\left(Z\left\{\dfrac{v-h}{\sigma + (1-\sigma)v}\right\}\right), & v \geq T_{ign} + 2\delta, \\ \exp\left(Z\left\{\dfrac{v-h}{\sigma + (1-\sigma)v}\right\}\right) H^\delta(v - T_{ign} - \delta), \\ \quad T_{ign} \leq v < T_{ign} + 2\delta, \\ 0, \quad v < T_{ign}, \end{cases} \tag{6}$$

where

$$H^\delta(x) = \begin{cases} \dfrac{1}{1 + e^{\frac{4x\delta}{\delta^2 - x^2}}}, & \text{for } |x| < \delta, \\ 1, & \text{for } x \geq \delta, \\ 0 & \text{for } x \leq -\delta, \end{cases}$$

or some other smooth approximation of the Heaviside function H. In other words, H^δ is a function such that in the distributional sense $\lim_{\delta\to 0^+} H^\delta = H$, and $H^\delta(x) = 1$, for $x > \delta$, $H^\delta(x) = 0$, for $x < -\delta$.

For numerical computations in Sections 3 and 5, we choose δ small enough so that the front velocity in the system with F_δ is close to the velocity in the system with F_d.

It is known [9] that for the system with $\epsilon = 0$ the front solution with $F = F_\delta$ converges as $\delta \to 0^+$ to the front of the system with the reaction rate given by F_d. The situation is more complicated when $\epsilon > 0$. The fronts in the $\epsilon > 0$ case exist for any $\delta \geq 0$. The proof is by construction and is based on geometric singular perturbation theory which guarantees continuity in δ as long as F_δ is smooth, i.e. for $\delta > 0$. The existing analytic proof