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Theory of singular vortex solutions of the nonlinear Schrödinger equation

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Abstract

We present a systematic study of singular vortex solutions of the critical and supercritical two-dimensional nonlinear Schrödinger equation. In particular, we study the critical power for collapse and the asymptotic blowup profile of singular vortices.

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1. Introduction

The focusing d-dimensional nonlinear Schrödinger equation (NLS)

$$i\psi_t(t,\mathbf{x}) + \Delta\psi + |\psi|^{2\sigma}\psi = 0, \quad \psi(0,\mathbf{x}) = \psi_0(\mathbf{x}), \tag{1}$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_d)$ and $\Delta = \partial_{x_1x_1} + \cdots + \partial_{x_dx_d}$, is one of the canonical nonlinear equations in physics, arising in various fields such as nonlinear optics, plasma physics, Bose–Einstein condensates (BEC), and surface waves. The NLS (1) is called *subcritical* if $\sigma d < 2$. In this case, all solutions exist globally. In contrast, solutions of the *critical* ($\sigma d = 2$) and *supercritical* ($\sigma d > 2$) NLS can become singular in finite time $0 < T_c < \infty$, i.e., $\lim_{t\to T_c} \|\psi\|_{H^1} = \infty$, where $\|\psi\|_{H^1} = \sqrt{\int |\psi|^2\,\mathrm{d}\mathbf{x} + \int |\nabla\psi|^2\,\mathrm{d}\mathbf{x}}$. See, e.g., [1] for more information.

In this study we consider singular solutions of the two-dimensional NLS, which in polar coordinates is given by

$$i\psi_t(t,r,\theta) + \psi_{rr} + \frac{1}{r}\psi_r + \frac{1}{r^2}\psi_{\theta\theta} + |\psi|^{2\sigma}\psi = 0, \quad \psi(0,r,\theta) = \psi_0(r,\theta). \tag{2}$$

This equation is critical when $\sigma = 1$ and supercritical when $\sigma > 1$. We focus on *vortex solutions* of the form

$$\psi(t,r,\theta) = A(t,r)e^{im\theta}, \quad m \in \mathbb{Z}. \tag{3}$$

It is relatively easy to produce optical vortices experimentally. As a result, vortices have been intensively studied, both theoretically and experimentally, in the nonlinear optics literature. More recently, vortex solutions have been studied, both theoretically and

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experimentally, in Bose–Einstein condensates (BEC).[1] However, almost all of this research effort has been on non-collapsing vortices. In fact, to the best of our knowledge, the only studies of collapsing vortex solutions are those of Kruglov and co-workers [3, 4] and of Vuong et al. [5]. Therefore, there is a huge gap between the available theory on non-vortex and vortex singular NLS solutions.

In this study, we present a systematic study of singular vortex solutions of the critical and supercritical NLS (2). In particular, we ask to what extent the available theory for singular NLS solutions remains valid for the subset of singular vortex solutions. Of course, all the rigorous results that were previously derived for singular NLS solutions remain valid for the special case of vortex solutions. As we shall see, however, in some cases stronger results can be obtained for collapsing vortices (e.g., the critical power for collapse). In addition, we find that some of the non-rigorous results for singular non-vortex solutions that were derived using asymptotic analysis and numerical simulations (e.g., stability of blowup profiles) do change for vortex solutions. Intuitively, the main reason for this qualitative difference is that vortex solutions must vanish at the origin, where the phase is undefined. Therefore, singular vortex solutions that collapse at the origin are identically zero there and must have a ring profile. This is very different from non-vortex singular solutions, whose amplitude at the collapse point increases to infinity as they collapse, regardless of whether their peak value is at the collapse point (i.e., peak-type solution) or not (i.e., ring-type solution).

The paper is organized as follows. In Section 2, we recall the conservation laws of the NLS (2). In Section 3, we consider stationary vortex solutions of the form $\psi_{m,k}^{\text{stationary}} = R_{m,k}(r)\mathrm{e}^{\mathrm{i}\lambda t + \mathrm{i}m\theta}$.

In Section 4, we systematically study vortex solutions of the *critical* ($\sigma = 1$) two-dimensional NLS

$$\mathrm{i}\psi_t(t,r,\theta) + \psi_{rr} + \frac{1}{r}\psi_r + \frac{1}{r^2}\psi_{\theta\theta} + |\psi|^2\psi = 0, \quad \psi(0,r,\theta) = \psi_0(r,\theta). \tag{4}$$

In Section 4.1, we study the profiles $R_{m,k}(r)$ of the stationary vortex solutions and in particular the ground state profile $R_{m,0}$. In Section 4.2 we show that as in the vortex-free case, there are two types of explicit blowup solutions of the critical NLS (4): $\psi_{R_m}^{\text{explicit}}$ with a linear blowup rate which are in H^1, and $\psi_{G_m}^{\text{explicit}}$ with a square-root blowup rate which are not is H^1. However, while in the vortex-free case, $\psi_{R_0}^{\text{explicit}}$ is a peak-type solution and $\psi_{G_0}^{\text{explicit}}$ is a ring-type solution, in the vortex case, both $\psi_{R_m}^{\text{explicit}}$ and $\psi_{G_m}^{\text{explicit}}$ are ring-type solutions. Moreover, unlike $\psi_{G_0}^{\text{explicit}}$, these *singular vortex ring solutions are identically zero at the singularity point* $r = 0$. In Section 4.3 we consider the critical power (L^2 norm) for collapse in the critical NLS (4). Recall that in the non-vortex case the critical power is equal to $P_{\text{cr}} = \int |R_{0,0}|^2\, r\mathrm{d}r$, where $R_{0,0}$ is the ground state solution of

$$R'' + \frac{1}{r}R' - R + R^3 = 0, \qquad R'(0) = 0, \qquad R(\infty) = 0. \tag{5}$$

In contrast, the critical power for radially-symmetric vortex initial conditions of the form $\psi_0 = A_0(r)\mathrm{e}^{\mathrm{i}m\theta}$ is $P_{\text{cr}}(m) = \int |R_{m,0}|^2\, r\mathrm{d}r$, where $R_{m,0}$ is the ground state solution of

$$R_m''(r) + \frac{1}{r}R_m' - \left(1 + \frac{m^2}{r^2}\right)R_m + R_m^3 = 0, \qquad R_m'(0) = 0, \qquad R_m(\infty) = 0.$$

The critical power $P_{\text{cr}}(m)$ increases with m, and is approximately given by $P_{\text{cr}}(m) \approx 4\sqrt{3}m$. In particular, it is significantly larger than $P_{\text{cr}} := P_{\text{cr}}(m = 0) \approx 1.86$. In [3], Kruglov and Logvin estimated the value of $P_{\text{cr}}(m)$ by assuming that the vortex solution collapses with a self-similar Laguerre-Gaussian profile. We show that this estimate is a crude upper bound, and that this is due to the use of the aberrationless approximation and the fact that the Laguerre-Gaussian profile does not provide a good approximation of $R_{m,0}$. In addition, we provide a simple criterion to determine whether an initial profile is "close" to $R_{m,0}$, in which case the excess power above $P_{\text{cr}}(m)$ needed for collapse is "small". We then ask what is the critical power when the initial vortex profile is not radially-symmetric, e.g, when $\psi_0 = A_0(x,y)\mathrm{e}^{\mathrm{i}m\theta}$ where A_0 is real but not symmetric in r. In Section 4.3.3 we show that in this case, the vortex solution can collapse with power *below* $P_{\text{cr}}(m)$ but, of course, above P_{cr}. This is exactly opposite from the non-vortex case, in which deviations from radial symmetry increase the threshold power for collapse [6]. The reason for this difference is as follows. In the vortex-free case, the $\psi_{R_{0,0}}$ profile is stable under symmetry-breaking perturbations. In contrast, vortices are unstable under symmetry-breaking perturbations and, when perturbed azimuthally, they break into a ring of filaments. Since these filaments do not collapse at $r = 0$, the vorticity does not prohibit them from collapsing with the $R_{0,0}$ profile. Hence, the critical power for collapse of each of the filaments is $P_{\text{cr}} = P_{\text{cr}}(m = 0)$. In Section 4.4 we show that as in the vortex-free case, all stationary vortex solutions are strongly unstable. In Section 4.5 we show that as in the vortex-free case, the explicit vortex blowup solution $\psi_{R_{m,0}}^{\text{explicit}}$ is unstable.

Section 4.6 is devoted to the study of the asymptotic blowup profiles of critical vortex solutions. In [7], Merle and Raphael proved that all singular solutions of the critical NLS (4) with power is slightly above P_{cr} collapse with the asymptotic $\psi_{R_{0,0}}$ profile and that

[1] For a recent review on vortices in Optics and in BEC, see [2].