Available online at www.sciencedirect.com
ScienceDirect
ELSEVIER
Physica D 237 (2008) 2731–2760
PHYSICA D
www.elsevier.com/locate/physd

Stability and instability of nonlinear defect states in the coupled mode equations—Analytical and numerical study

Roy H. Goodman[a,*], Michael I. Weinstein[b]

[a] *Department of Mathematical Sciences, New Jersey Institute of Technology, Newark, NJ 01710, United States*
[b] *Department of Applied Physics and Applied Mathematics, Columbia University, New York, NY 10027, United States*

Received 31 August 2007; received in revised form 7 April 2008; accepted 15 April 2008
Available online 24 April 2008

Communicated by J. Bronski

Abstract

Coupled backward and forward wave amplitudes of an electromagnetic field propagating in a periodic and nonlinear medium at Bragg resonance are governed by the nonlinear coupled mode equations (NLCME). This system of PDEs, similar in structure to the Dirac equations, has gap soliton solutions that travel at any speed between 0 and the speed of light. A recently considered strategy for spatial trapping or capture of gap optical soliton light pulses is based on the appropriate design of localized defects in the periodic structure. Localized defects in the periodic structure give rise to defect modes, which persist as *nonlinear defect modes* as the amplitude is increased. Soliton trapping is the transfer of incoming soliton energy to *nonlinear* defect modes. To serve as targets for such energy transfer, nonlinear defect modes must be stable. We therefore investigate the stability of nonlinear defect modes. Resonance among discrete localized modes and radiation modes plays a role in the mechanism for stability and instability, in a manner analogous to the nonlinear Schrödinger/Gross–Pitaevskii (NLS/GP) equation. However, the nature of instabilities and how energy is exchanged among modes is considerably more complicated than for NLS/GP due, in part, to a continuous spectrum of radiation modes which is unbounded above and below. In this paper we (a) establish the instability of branches of nonlinear defect states which, for vanishing amplitude, have a linearization with eigenvalues embedded within the continuous spectrum, (b) numerically compute, using Evans function, the linearized spectrum of nonlinear defect states of an interesting multiparameter family of defects, and (c) perform direct time-dependent numerical simulations in which we observe the exchange of energy among discrete and continuum modes.

Keywords: Nonlinear optics; Defects; Stability; Gap solitons; Evans function

1. Introduction

In optical fiber communication systems, data is carried as pulses of light. Expensive and rate-limiting steps in these systems come in processing the data at so-called optical/electrical interfaces. Processing is typically done electronically; signals are converted to electronic form, read, processed, and retransmitted. A major goal, therefore, is to bypass the optical/electrical interface and implement "all-optical" processing by using the nonlinear optical properties of the medium. Hence, there has been great interest in finding novel materials and optical structures (arrangements of materials) which effect light in different ways. Among these are Bragg gratings and Bragg gratings with defects–one dimensional arrangements, as well as higher-dimensional structures such as photonic crystal fibers in which the grating structure is transverse to the direction of propagation [43], and other two and three-dimensional structures [16].

In an optical fiber Bragg grating, the refractive index of the glass varies periodically, with period resonant with the carrier wavelength of the propagating light; see Fig. 1.1. Light propagation through such fibers has several interesting properties that

* Corresponding author. Tel.: +1 973 642 4261.
E-mail address: roy.goodman@njit.edu (R.H. Goodman).

0167-2789/$ - see front matter © 2008 Published by Elsevier B.V.
doi:10.1016/j.physd.2008.04.009

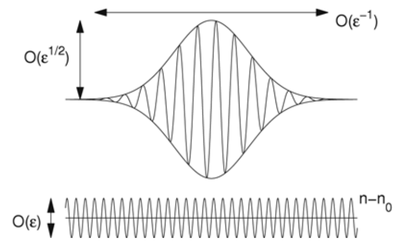

Fig. 1.1. Schematic of Bragg resonance condition. Periodic refractive index (lower figure) and electric field envelope with carrier wave in Bragg resonance ($\lambda = 2 \times$ period).

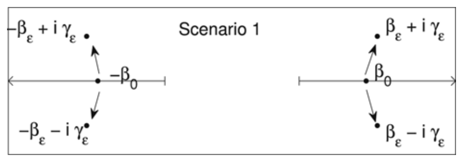

Fig. 1.2. Instability onset scenario 1: as the amplitude of the nonlinear defect mode is increased from 0 to ϵ the embedded frequencies $\pm\beta_0$ split into complex frequencies $\pm\beta_\epsilon \pm \mathrm{i}\gamma_\epsilon$.

makes them useful in technological applications. The grating structure couples forward-moving light at the resonant wavelength to backward-moving light of the same wavelength. In the low-amplitude (linear) limit, this makes the fiber opaque to light in a certain range of wavelengths, the so-called photonic band-gap. When the amplitude of the light is increased, the band-gap is shifted due to the nonlinear dependence of refractive index on intensity. Thus, a range of wavelengths that are non-propagating at low intensity are shifted into the range of propagating wavelengths (the pass-band) at higher intensities. This is the mechanism by which localized pulses known as gap solitons exist; see Section 2.2. In the regime of weak nonlinearity and Bragg resonance, Maxwell's equations can be reduced, via multiple scale asymptotic methods, to the nonlinear coupled mode equations (NLCME), reviewed in Section 2.1 [2,15,19,32].[1]

Gap solitons may, in theory, propagate (in the stationary reference frame) at any speed between 0 and c/n. Here, c denotes the vacuum speed of light, n the refractive index of the optical fiber core and c/n is the speed of light in the fiber without the grating. In [29], we showed via modeling, analysis and numerical simulation, that propagating optical gap solitons, could be trapped at specially-constructed defects in the grating structure. Trapping and interaction of gap solitons in a number of related structures is considered in [14,46,47]. Recent experimental advances have made possible the slowing of propagating gap soliton light pulses from $0.5 \times c$ [23] to $0.16 \times c$ [52].

The focus of the present paper is on the stability and dynamics of such trapped light. Localized defects in a grating appear as spatially localized *potentials* in the NLCME model. In the linear (low light intensity) limit, the resulting *linear coupled mode equations* with potentials have spatially localized *linear defect eigenstates*; see Section 2.3. As the intensity is increased from zero, *nonlinear defect modes*, "pinned" at the defect location, bifurcate from the zero state at the linear eigenfrequencies; see Section 2.4.

Trapping of a gap soliton by a defect can be understood as the resonant transfer of energy from an incoming soliton to a pinned nonlinear defect mode. In [29] we demonstrated through numerical experiments such resonant energy transfer/trapping, for sufficiently slow soliton pulses. The analogous question has also been studied for the nonlinear Schrödinger/Gross–Pitaevskii (NLS/GP) equation; see [28,34–36] and references cited therein. For the purpose of comparison, we review the stability of and interactions between nonlinear defect modes of NLS/GP in Section 3.

In order for the energy localized in a defect to remain spatially confined in a nonlinear defect mode, it is necessary that the mode be stable. Thus, in this paper we consider the stability of nonlinear defect modes for a large class of defects introduced in [29] by

- studying the linearized spectral problem about different families (branches in the global bifurcation diagram) of nonlinear defect modes; see Sections 4 and 5 for analytical perturbation theory and numerics, and
- studying time-dependent simulations of the initial value problem for NLCME. We consider defects which support multiple nonlinear linear defect modes. As the time-evolution proceeds these nonlinear bound state families compete for the energy localized in the defect; see Section 6.

The linearized stability of a nonlinear defect mode is governed by a spectral problem of the form:

$$\Sigma_3 H \psi = \beta\, \psi, \qquad \Sigma_3^* = -\Sigma_3, \quad H^* = H; \tag{1.1}$$

see Section 4. Eigenvalues are complex values of β, for which (1.1) has a nontrivial L^2 solution, giving rise to a time-dependent solution of the linearized dynamics $\phi = \psi \mathrm{e}^{-\mathrm{i}\beta t}$. The self-adjoint operator, H, can be expressed as $H = H_0 + W$, where H_0

[1] A special case of NLCME (Eq. (2.1) with the self-phase modulation terms omitted) is the massive Thirring system [25], a completely integrable PDE modeling the interaction of massive fermions. The methods described in this article apply equally well to the massive Thirring system.