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# High-order control for symplectic maps

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## HIGHLIGHTS

- Control for symplectic maps in a neighborhood of an elliptic equilibria.
- Normal form for maps using Lie transforms.
- Effective numerical algorithm for the control problem for maps.
- Enlarging the size of the stability domain of a 2D Hénon type map.

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ABSTRACT

We revisit the problem of introducing an *a priori control* for devices that can be modeled via a symplectic map in a neighborhood of an elliptic equilibrium. Using a technique based on Lie transform methods we produce a normal form algorithm that avoids the usual step of interpolating the map with a flow. The formal algorithm is completed with quantitative estimates that bring into evidence the asymptotic character of the normal form transformation. Then we perform an heuristic analysis of the dynamical behavior of the map using the invariant function for the normalized map. Finally, we discuss how control terms of different orders may be introduced so as to increase the size of the stable domain of the map. The numerical examples are worked out on a two dimensional map of Hénon type.



## 1. Introduction

To control engineered human devices is a necessity to ensure their right behavior, that is the system should follow as close as possible the wanted trajectory, regardless from any deviations due to noise and possible errors. For this reason the so called *problem of control* of dynamical systems is a long lasting research field where engineers, physicists and mathematicians, among others, have been very active.

Nowadays this represents a whole field in applied mathematics and in engineering with sub-fields such as control and systems [1], linear systems [2] or optimal control [3]. Because of the advancement of technology, controllers are in action almost everywhere in our daily routine, from the design of the water tank of the ordinary flush toilet to the design of a lateral and longitudinal control of a Boeing (and thus the autopilot) or in the satellites attitude control.

The common feature of most of the above quoted theories lies in the idea of *feedback*, that is one should access to the actual state of the system (or to any relevant measure of it) and then act on the system to achieve the desired goal, usually to stabilize the trajectory around the nominal one. The controller is thus switched on and off when required *during* the evolution of the system. This implicitly means that the typical time scale of the system dynamics is several order of magnitude bigger than the time required to the controller to read the state of the system, to compute the correction and eventually modify the system accordingly.

A procedure based on feedback is clearly a hard task, if not impossible, when one deals with very complex devices such as, e.g., a particle accelerator, mainly because the previous assumption about the time scale is no longer satisfied: no control could react as fast as the particles moving at almost the speed of light. In this case however a control can be obtained by some external tuning to be performed *before* the device is turned on. We hereby develop this idea by presenting a method that can be applied in case a reliable mathematical model of the device is available. The basis of the method is to introduce high order controllers without the need of switches and based on the identification of *dangerous terms*, that

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once removed – or they impact reduced – will allow to achieve the desired goal; the method is thus, in some sense, based on an *a priori* analysis.

The above idea has been exploited already in the end of the eighties: see the volume [4]. Recently the method has been widely investigated on the basis of the ideas presented by Vittot [5] in the continuous time case and then adapted to the discrete time case, namely maps (see, e.g., [6–8]). The introduction and the intensive use of the Lie transform methods are the key ingredient to easily obtain high order controllers.

For the sake of clarity we consider the problem of control in the Hamiltonian framework, let us stress however that the following ideas can be straightforwardly extended beyond it. In its simplest formulation it can thus be stated as follows. Consider a nearly integrable canonical system of differential equations in the neighborhood of an elliptic equilibrium, as described by the Hamiltonian

$$H(x,y) = H_0(x,y) + F(x,y), \qquad (1)$$

with

$$H_0(x,y) = \frac{1}{2}\sum_{l=1}^{n} \omega_l (x_l^2 + y_l^2)$$

where $(x, y) \in \mathbb{R}^{2n}$ are the canonically conjugate coordinates, $\omega \in \mathbb{R}^n$ are the frequencies and $F(x, y)$ is either a power series or a polynomial of finite degree starting with terms of degree at least 3. The problem is to find a function $G(x, y)$ such that the Hamiltonian $H_0 + F - G$ can be put in a simpler form, in most applications this would mean to conjugate it to an integrable one by a canonical transformation. The answer should be non-trivial, in the sense that the obvious choice $G = F$ is not accepted, because the nonlinear character of the system should be preserved in some form. E.g., one looks for a nonlinear integrable Hamiltonian. It is rather requested that $G$ should be smaller than $F$, e.g., in some norm.

A very similar problem is concerned with symplectic maps in a neighborhood of an elliptic equilibrium. One considers a map of a neighborhood of the origin in the plane $\mathbb{R}^2$ that is written as

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \Lambda_\omega \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix} \qquad (2)$$

where $\Lambda_\omega$ is a suitable rotation matrix (see formula (12)), while $f(x, y)$ and $g(x, y)$ are either power series or polynomials starting with terms at least of degree 2, and are required to satisfy the symplecticity conditions. Again the problem is to add a non-trivial control term such that the resulting modified map is conjugated to a rotation, possibly a twist one.

In the present paper we revisit the problem of control referring in particular to the case of maps. We reformulate it using the tool of Lie transforms, and point out different ways of introducing control terms.

It should be noticed that the problem of control is essentially a particular formulation of the classical ''general problem of dynamics'', so named by Poincaré (see [9], Vol. I, §13). That is, to investigate the dynamics of a Hamiltonian system

$$H(p,q) = h(p) + \epsilon f(p,q;\epsilon) \qquad (3)$$

where $(p, q) \in \mathcal{G} \times \mathbb{T}^n$ with $\mathcal{G} \subset \mathbb{R}^n$ are action-angle variables and $\epsilon$ is a small parameter. The Hamiltonian is assumed to be holomorphic in all variables and in the parameter. As proved by Poincaré himself, the system is generically non-integrable due to the extreme complexity of the orbits. However some insight on the dynamics of the system is available by searching for weaker properties than complete integrability. Just to quote some results that came after Poincaré, we mention the existence of invariant tori [10], the theory of exponential stability [11–16], the super-exponential stability [17,18]. After Poincaré's work it was soon remarked that the problem is substantially simplified if one considers the case of an elliptic equilibrium, i.e., the Hamiltonian (1) (see [19–22]). Extensions to the case of maps have been studied by Poincaré, who introduced the idea of reducing the flow of differential equations to a map [23,24]. On the other hand, Birkhoff exploited the idea of interpolating an area preserving map of the plane by a Hamiltonian flow [25].

A wide discussion of the problem of control for Hamiltonian systems in the light of renormalization theory has been made by Gallavotti [26], who calls *counterterms* the control terms. A constructive approach has been proposed by Vittot [5], by using the Lie series formalism in order to construct a suitable normal form. Vittot's ideas have been exploited in order to find control terms that may reduce chaos (or increase the size of the stability region around the equilibrium point) in models of interest in physics, e.g., the dynamics of magnetized plasmas [27–30]. Although a full control in the sense initially proposed by Vittot is unrealistic in a practical application (as we shall discuss below), introducing some properly chosen terms in the Hamiltonian may significantly reduce chaos, thus stabilizing the dynamics. This aspect, which may present a considerable practical interest, is discussed in the above quoted papers, with explicit examples.

Concerning maps, the problem has been widely studied since the eighties of the past century in a series of papers by a group of authors including Bazzani, Giovannozzi, Servizi, Todesco and Turchetti. They investigated the normal form for symplectic maps in view of application to betatronic motions in accelerators [31–34].

However, we think that a better insight on the problem may be achieved by introducing some changes both in the technical tools and in the theoretical framework. This is what we are going to discuss in this paper, making reference in particular to the case of a (symplectic) map in the neighborhood of an elliptic equilibrium.

The paper is organized as follows. In the rest of the present section we include an informal discussion of the technical improvement and of the theoretical framework. In Section 2 we present the formal algorithm that allows us to calculate the normal form of the map and of different forms of the control terms. In Section 3 we work out all the quantitative estimates on the normal form and on the possible control terms. The numerical application is presented in Section 4, where we illustrate an heuristic method to predict the size of the stable region using the normal form and compare the results with a direct iteration of the map.

### 1.1. Technical improvements

The construction of a normal form for symplectic maps is usually worked out by using interpolation via a canonical system of differential equations, as proposed by Birkhoff. The reason is perhaps that for the Hamiltonian case there are several methods available, the most effective ones being based on performing canonical transformation using the method of Lie series, as proposed in the recent paper of Vittot [5].

Let us start by making two remarks. The first one is that going through a flow of differential equations in order to represent a map is a lengthy procedure, that may be desirable to avoid. The second remark is that replacing the method of Lie series with that of Lie transforms makes the calculations (to be done via algebraic manipulation) definitely more effective and introduces a lot of flexibility.

We adopt here a language that is common in the milieu of Celestial Mechanics. Given a vector field $X(x)$ on a $n$-dimensional manifold with coordinates $x$ the Lie series is introduced as the differential operator $\exp\big(tL_X\big) = \sum_{k\geq 0} \frac{t^k L_X^k}{k!}$, where $L_X$ is the Lie derivative along the flow generated by the vector field $X$ and $t$ is the time parameter of the flow. In this form the Lie series