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Phenomenological model for predicting stationary and non-stationary spectra of wave turbulence in vibrating plates

T. Humbert [a,*], C. Josserand [b], C. Touzé [c], O. Cadot [c]

[a] SPEC, CNRS, CEA, Université Paris-Saclay, 91191 Gif-sur-Yvette, France

[b] Institut d'Alembert, CNRS, UMR 7190, Sorbonne Universités, UPMC, 75005 Paris, France

[c] IMSIA, ENSTA ParisTech, CNRS, CEA, EDF, Université Paris-Saclay, 828 bd des Maréchaux, 91762 Palaiseau cedex, France

HIGHLIGHTS

- A phenomenological model has been derived in the case of elastic vibrating plates.
- Without damping, self-similar dynamics for forced and free turbulence are retrieved.
- In the framework of damped wave turbulence, self-similar universal solutions are given.
- An equation that links power spectra, damping law and injected power has been found.
- The agreement of the model with experiments and simulations is found to be good.

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ABSTRACT

A phenomenological model describing the time-frequency dependence of the power spectrum of thin plates vibrating in a wave turbulence regime, is introduced. The model equation contains as basic solutions the Rayleigh–Jeans equipartition of energy, as well as the Kolmogorov–Zakharov spectrum of wave turbulence. In the Wave Turbulence Theory framework, the model is used to investigate the self-similar, non-stationary solutions of forced and free turbulent vibrations. Frequency-dependent damping laws can easily be accounted for. Their effects on the characteristics of the stationary spectra of turbulence are then investigated. Thanks to this analysis, self-similar universal solutions are given, relating the power spectrum to both the injected power and the damping law.

1. Introduction

The Wave (or Weak) Turbulence Theory (WTT) aims at describing the long-term behaviour of weakly nonlinear systems where the nonlinearity controls the exchanges between scales [1–3]. Under classical assumptions such as dispersivity, weak nonlinearities and the existence of a transparency window in which the dynamics is assumed to be conservative, a kinetic equation can be deduced for the slow dynamics of the spectral amplitude. In addition to the Rayleigh–Jeans spectrum that corresponds to the equipartition of the conserved quantity, here the energy, a broadband Kolmogorov–Zakharov (KZ) spectrum of constant energy flux is predicted, by analogy with hydrodynamic turbulence [1,2]. Such dynamics has been firstly studied for ocean (gravity) waves [4–6] and since then in systems such as capillary waves [7,8], nonlinear optics [9] or plasmas [10].

A wave turbulence spectrum for elastic vibrating plates has been deduced theoretically and observed numerically in [11]. The theoretical analysis considers the dynamics of a geometrically nonlinear thin vibrating plate in the framework of the Föppl–von Kármán (FVK) equations. The WTT analysis leads to the prediction of a direct cascade characterized by a KZ spectrum with constant energy flux. Soon after, two independent experiments performed on thin elastic plates [12–14] did not recover the theoretically predicted and numerically observed spectra, questioning the validity of the underlying assumptions of WTT in the case of vibrating plates. Recently, an experimental and numerical study considering the effect of damping on the turbulent properties of thin vibrating plates has clearly established that [15]:

* Corresponding author. Tel.: +33 676 01 31 41.
E-mail address: thomas.humbert@cea.fr (T. Humbert).

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- In experiments, damping acts at all scales such that the assumption of a transparency window, a domain in the wave number space where dissipation and injection can be neglected, is questionable.
- Modifying the damping alters the shape of the velocity power spectra so that a direct comparison with the predicted spectra is out of reach in experimental conditions.
- However, by including the experimentally measured damping laws in the numerical simulations of the full dynamics (the FVK equations), a good agreement with the experiments is retrieved. This suggests that the discrepancies between the experiments and the WTT predictions are mainly due to damping.

These conclusions have been corroborated by a numerical study where the damping was gradually modified, from the experimentally measured law to a vanishing value in a given frequency band [16], showing also how the spectra are modified by a small yet non-negligible values of damping found in real plates.

Accounting for dissipation within the WTT framework remains challenging since the analytic calculations are based on the long time asymptotic evolution of the weakly nonlinear Hamiltonian dynamics. The injection and dissipation in this context can be seen as boundary conditions imposed to the transparency window in the wave number space and to the best of our knowledge, we do not know any analytical attempt to introduce dissipation within the WTT. Another option would be to find an alternative description of the dynamics of the power spectrum, where adding dissipation appears more straightforward. The alternative can be provided by using a phenomenological model describing the temporal evolution of the power spectra, as first proposed by Leith for hydrodynamic isotropic turbulence [17]. These models provide a natural framework for investigating unsteady and self-similar dynamics in a variety of context [17–22]. They are generally derived from ad-hoc assumptions, by constructing a model equation admitting as stationary solutions both the Rayleigh–Jeans equipartition of energy and the KZ spectrum. This results in a nonlinear diffusion equation in the wave number (k-space) or the frequency (ω-space) domain, which mimics the energy transfer within the modes. Thanks to this approach, ideal situations can be investigated, as for instance the injection of a constant flux of energy at small scales and its diffusion, or the evolution of an initial condition in absence of dissipation. Self-similar dynamics are generally observed in these cases.

The goal of this paper is thus to derive and investigate such a phenomenological model in the case of elastic vibrating plates. The model equation should contain both Rayleigh–Jeans and KZ solutions. Injection and dissipation terms are then introduced in order to study more particularly the effects of the damping. Two main results are obtained. First, self-similar dynamics for forced and isolated turbulence in the absence of dissipation are retrieved. In a second part, the effect of the damping on the cascading turbulent spectrum is investigated, exhibiting a self-similar solution relating the power spectrum to the injected power and the damping law.

2. Model equation

The application of the wave turbulence theory to the Föppl–von Kármán thin plate equations has been performed in [11] (see Appendix A for the dimensional and non-dimensional forms of these equations. Note that for this section, all values are dimensionless). Without recalling the details of the derivation and the complex form of the kinetic equation, one only needs to remind that the two stationary solutions of the kinetic equation, written here under the form of a density of energy E_ω, function of the frequency ω, are:

- The Rayleigh–Jeans equilibrium solution, where the energy E_ω is equally parted along all the available modes. Consequently, the density of energy E_ω is a constant that is denoted as C:

$$E_\omega = C. \tag{1}$$

- The Kolmogorov–Zakharov solution, for which an energy flux ε is transferred along the cascade until its dissipation near $\omega^\star$, the cut-off frequency of the spectrum. Referring to [11], the energy spectrum in this case is such that

$$E_\omega^{KZ} = A\varepsilon^{\frac{1}{3}} \log^{\frac{1}{3}}\left(\frac{\omega^\star}{\omega}\right), \tag{2}$$

where A is a constant. The specific form of this solution, consisting in a logarithmic correction of the Rayleigh–Jeans spectrum, comes from a degeneracy of the equilibrium solution in a similar manner as for the nonlinear Schrödinger equation [9]. In fact, this logarithmic correction is obtained using a perturbative expansion and is valid far from $\omega^\star$. Therefore, although Eq. (2) exhibits a steep cut-off because of the non-existence of the mathematical solution above $\omega^\star$ (negative energy), experiments and numerical simulations do not show such a behaviour, and the spectrum decreases more smoothly as ω increases in the vicinity of $\omega^\star$ [15,23,24].

The phenomenological model is directly deduced from these stationary solutions of the energy spectrum. Let us consider the following diffusion-like equation in the ω-space for the energy spectrum $E_\omega(\omega, t)$:

$$\partial_t E_\omega = \partial_\omega(\omega E_\omega^2 \partial_\omega E_\omega), \tag{3}$$

where ∂_t and ∂_ω refer respectively for the partial derivatives with respect to time and angular frequency. The energy flux associated to this equation reads straightforwardly

$$\varepsilon = -\omega E_\omega^2 \partial_\omega E_\omega. \tag{4}$$

Thanks to the identification of the energy flux ε, the proportionality constant A of Eq. (2) is then uniquely defined as $A = 3^{\frac{1}{3}}$. Hence, for the phenomenological model the KZ solution finally reads:

$$E_\omega^{KZ} = (3\varepsilon)^{\frac{1}{3}} \log^{\frac{1}{3}}\left(\frac{\omega^\star}{\omega}\right). \tag{5}$$

The model equation, Eq. (3), is constructed so that Eqs. (1) and (2) are stationary solutions ($\partial_t E_\omega = 0$). The Rayleigh–Jeans equilibrium is a trivial solution to Eq. (3) in the stationary case since $\partial_\omega E_\omega = 0$. For the KZ spectrum, one has just to verify, by deriving Eq. (2) with respect to ω, that $\omega E_\omega^2 \partial_\omega E_\omega$ is constant with respect to ω. Because this model equation has been deduced in the dimensionless framework, only a numerical prefactor, which could be easily absorbed by a rescaling of the time, should be present on the right-hand side of Eq. (3).

The phenomenological equation is nothing else than a nonlinear diffusion equation in the frequency space, in the spirit of the Richardson cascade view of turbulent processes [25]. However, a direct derivation of this equation starting from the kinetic equation cannot be done formally, and only qualitative arguments can be deduced from a local approach on the kinetic equation [1] (Section 4.3). In fact, attempts to deduce such simplified Fokker–Planck equation from the weak turbulence equations go back to the pioneering works done for ocean waves by Hasselmann [26–28], although additional approximations were needed to deduce such local models in frequency.

Nonlinear diffusion equations can exhibit important differences as compared to the linear diffusion one. In particular, singularity can be formed by the nonlinear dynamics and compact support solutions can also be present, by opposition to the linear diffusion