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ScienceDirect
Physica D 237 (2008) 1912–1920
PHYSICA D
www.elsevier.com/locate/physd

3D Euler about a 2D symmetry plane

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Available online 15 February 2008

Abstract

Initial results from new calculations of interacting anti-parallel Euler vortices are presented with the objective of understanding the origins of singular scaling presented by Kerr [R.M. Kerr, Evidence for a singularity of the three-dimensional, incompressible Euler equations, Phys. Fluids 5 (1993) 1725–1746] and the lack thereof by Hou and Li [T.Y. Hou, R. Li, Dynamic depletion of vortex stretching and non-blowup of the 3-D incompressible Euler equations, J. Nonlinear Sci. 16 (2006) 639–664]. Core profiles designed to reproduce the two results are presented, new more robust analysis is proposed, and new criteria for when calculations should be terminated are introduced and compared with classical resolution studies and spectral convergence tests. Most of the analysis is on a $512\times128\times2048$ mesh, with new analysis on a just completed $1024\times256\times2048$ used to confirm trends. One might hypothesize that there is a finite-time singularity with enstrophy growth like $\Omega \sim (T_c - t)^{-\gamma_\Omega}$ and vorticity growth like $\|\boldsymbol{\omega}\|_\infty \sim (T_c - t)^{-\gamma}$. The new analysis would then support $\gamma_\Omega \approx 1/2$ and $\gamma > 1$. These represent modifications of the conclusions of Kerr [op. cit.]. Issues that might arise at higher resolution are discussed.

PACS: 47.10.A-; 47.11.Kb; 47.15.ki

Keywords: Euler equations; Fluid singularities; Vortex dynamics

1. Introduction

One definition of solving Euler's 3D incompressible equations [1] is determining whether or not they dynamically generate a finite-time singularity if the initial conditions are smooth, in a bounded domain and have finite energy. The primary analytic constraint that must be satisfied [2] is:

$$\int_0^T \|\boldsymbol{\omega}\|_\infty \mathrm{d}t \to \infty \tag{1}$$

where $\|\boldsymbol{\omega}\|_\infty$ is the maximum of vorticity over all space. To date, [3] remains the only fully 3D simulation of Euler's equations with evidence for a singularity consistent with this and related constraints [4]. Growth of the enstrophy production and stretching along the vorticity, plus collapse of positions, supported this claim [3]. Additional weaker evidence related to blowup in velocity and collapsing scaling functions was presented later [5].

There is only weak numerical evidence supporting these claims [6,7]. In a recent paper, as described in one of the invited talks of this symposium, [8] found evidence that the above scenario failed at late times.

This contribution will first comment on four issues raised at the symposium, then present preliminary new results. The four issues are:

- How should spurious high-wavenumber energy in spectral methods be suppressed?
- What criteria should be used to determine when numerical errors are substantial?
- What effect do the initial conditions have on singular trends? A cleaner initial condition is proposed.
- We introduce a new approach for determining whether there is singular behavior of the primary properties and the associated scaling. This is applied to both new and old data.

All calculations will be in the following domain: $L_x \times L_y \times L_z = 4\pi \times 4\pi \times 2\pi$ with free-slip symmetries in y and z and periodic in x with up to $n_x \times n_y \times n_z = 1024 \times 256 \times 2048$ mesh points. Using these symmetries only one-half of one of the anti-parallel vortices needs to be simulated.

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doi:10.1016/j.physd.2008.02.007

The "symmetry" plane will be defined as xz free-slip symmetry through the maximum perturbation of the initial vortices and the "dividing" plane will be defined as the xy free-slip symmetry between the vortices.

2. How should spurious high-wavenumber energy in spectral methods be suppressed?

A generic difficulty in applying spectral methods to localized physical space phenomena is the accumulation of spurious high-wavenumber energy that leads to numerical errors.

What is the best approach for eliminating these spurious modes? We have compared the old-fashioned 2/3rds dealiasing versus the recently proposed 36th-power hyperviscous filter [8, 11]. Detailed tests to be described in a later paper show that the latter is better in the sense that for several quantities, such as the peak vorticity, lower resolution calculations follow the high-resolution cases longer. But a combination of the two approaches works even better, and that is what is used here.

Still, caution is required for any of these approaches as the hyperviscosity can dissipate small structures such as the anomalous negative vorticity in the squared-off profile below. Surprisingly the 36th-order hyperviscosity does not appear to produce the ghost vortices that are a known artifact of lower-order schemes.

3. What criteria should be used to determine when numerical errors are substantial?

There are traditionally two approaches to this problem, one emphasizing local quantities such as $\|\boldsymbol{\omega}\|_\infty$, and the other emphasizing global quantities such as the mean square vorticity or enstrophy. We use both.

3.1. Local quantities and resolution

To determine local resolution it is important to check the convergence of local quantities such as:

- The maximum of vorticity $\|\boldsymbol{\omega}\|_\infty$. The location of $\|\boldsymbol{\omega}\|_\infty$ will be defined as $\boldsymbol{x}_\infty$.
- The local stretching of vorticity

$$\alpha = \hat{\boldsymbol{\omega}}_i e_{ij} \hat{\boldsymbol{\omega}}_j \tag{2}$$

where $\hat{\boldsymbol{\omega}} = \boldsymbol{\omega}/|\omega|$ and $e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$.

Following earlier work [3,8], we use the criteria that $\boldsymbol{x}_\infty$ cannot be closer than 6 mesh points from the dividing plane.

3.2. Integral quantities

Examples of integral quantities we could monitor are: energy, circulation (which are in principle conserved), enstrophy and helicity (which are in principle changing).

(i) Energy is robustly conserved by spectral methods even when under-resolved and therefore is not a useful test. Convergence of the energy spectrum [8] is only a partial test because it neglects phase errors.

(ii) Circulation in the upper half of the symmetry plane (i.e., the $z > 0$ half of the xz-plane, which is perpendicular to the primary direction of vorticity y) is conserved. Circulation in the equivalent half of the dividing plane is also conserved. In all of the initial conditions considered here, it is initially zero and ideally should remain so. Therefore, the circulations of the symmetry and dividing planes, $\sigma_y = \int_{z>0} \omega_y(x,0,z,t)\mathrm{d}x\,\mathrm{d}z$ and $\sigma_z = \int_{y>0} \omega_z(x,y,0,t)\mathrm{d}x\,\mathrm{d}y$ were monitored.

We have found that serious depletion of σ_y is controlled by n_z and the time this begins is independent of the high-wavenumber filter. Once n_z is set, by convergence of $\|\boldsymbol{\omega}\|_\infty$, we find that there is good convergence if $n_x = n_z/2$ and $n_y = n_z/4$. A later paper will provide more details on these convergence tests. We will violate the condition on n_y at late times due to current memory restrictions.

Without the circulation test, it is difficult to draw conclusions about the late times in [8] where they claim to see divergence from the scaling of Kerr [3].

(iii) Enstrophy Ω grows in time, so one test is to check how it is balanced by its production Ω_p, which we determine directly. The enstrophy and its production are

$$\Omega = \int \mathrm{d}V \omega^2, \qquad \Omega_p = 2\int \mathrm{d}V \omega_i e_{ij} \omega_j. \tag{3}$$

(iv) Helicity grows within the quadrant simulated (not over the full anti-parallel geometry), but its production is determined by pressure which has not been calculated.

4. What is the effect of the initial conditions on the potentially singular behavior?

4.1. Earlier descriptions

As ambiguities in the earlier description of the initial condition of [3] led to differences in the initial condition of Hou and Li [8], the community needs a clear description of a reproducible, clean initial condition that yields the trends of Kerr [3]. Ideally, we want an initial condition whose vorticity is purely positive in the upper half of the symmetry plane, which following Kelvin's theorem will remain positive for all subsequent times. These steps were used [3] to massage the vortex profile in order to achieve this:

(A) The first step in creating the initial profile of the vorticity core is to use an explicit function where the value and all derivatives went smoothly to zero at a given radius. See references in [3] for earlier work that had used a similar profile. To this, a localized perturbation in its position in x was given [9].

(B) The second step is to remove high-wavenumber noise by applying a symmetric high-wavenumber filter of the form: $\exp(-a(k_x^2 + k_y^2 + k_z^2)^2)$. Kerr [10] showed the undesirable side-effects if this is not done. However, it has become apparent that the high-wavenumber filter is not sufficient.