Available online at www.sciencedirect.com

ScienceDirect

Physica D 237 (2008) 2210–2217

PHYSICA D

www.elsevier.com/locate/physd

Kinematic variational principle for motion of vortex rings

Y. Fukumoto[a,*], H.K. Moffatt[b]

[a] Graduate School of Mathematics and Mathematical Research Center for Industrial Technology, Kyushu University 33, Fukuoka 812-8581, Japan

[b] Department of Applied Mathematics and Theoretical Physics, Centre for Mathematical Science, University of Cambridge, Wilberforce Road, Cambridge CB3 0WA, UK

Available online 12 February 2008

Abstract

We show how the ideas of topology and variational principle, opened up by Euler, facilitate the calculation of motion of vortex rings. Kelvin–Benjamin's principle, as generalised to three dimensions, states that a steady distribution of vorticity, relative to a moving frame, is the state that maximizes the total kinetic energy, under the constraint of constant hydrodynamic impulse, on an iso-vortical sheet. By adapting this principle, combined with an asymptotic solution of the Euler equations, we make an extension of Fraenkel–Saffman's formula for the translation velocity of an axisymmetric vortex ring to third order in a small parameter, the ratio of the core radius to the ring radius. Saffman's formula for a viscous vortex ring is also extended to third order.

PACS: 47.10.A-; 47.10.ad; 47.15.ki; 47.32.C-

Keywords: Vortex ring; Translation velocity; Variational principle; Iso-vortical sheet

1. Introduction

Euler opened up the field of topology when he presented the solution to the Königsberg bridge problem in 1735 [1]. As "geometry of position" in the title signifies, Euler envisaged a new type of geometric problem in which distance is not relevant. In 1750, he discovered the polyhedral theorem on the Euler characteristic, a summation of alternately signed numbers of vertices, edges and faces of a polyhedron [2]. This theorem stands as the cornerstone of topology. Almost at the same time, the Euler equations for fluid flows were born.

Euler's 1757 paper [3] certainly overcame the limitation to irrotational velocity field, posed by Bernoulli, and accommodated vorticity. However a century passed before Helmholtz discovered the key to the heart of vortex motion that the vortex lines are frozen into the fluid [4]. Helmholtz' theorem implied that link and knot types of vortex lines remain unchanged throughout the flow evolution. This implication, along with the invariance of circulation, sparked, in Scotland, the construction of atom models by knotted vortex tubes. Inspired by the vortex atom theory, Tait attempted classification of knot and link types [5]. It took another century for the helicity to be discovered [6–9]. This topological invariant is tied with linkage and knottedness of vortex filaments [9]. More precisely, the helicity embodies the Călugăreanu invariant [10], a summation of the writhe and the twist, of a twisted flux tube [11].

The study of the motion of vortex rings started simultaneously with the birth of the field of vortex dynamics [4]. Extending Helmholtz' analysis, Kelvin obtained the formula for velocity of an axisymmetric vortex ring, steadily translating in an inviscid incompressible fluid of infinite extent, for a distribution of vorticity, in the core, proportional to the distance from the axis of symmetry. The assumption is made that the ring is very thin:

$$\varepsilon = \sigma / R_0 \ll 1, \tag{1}$$

where σ is the core radius and R_0 is the ring radius. The formula allowing for an arbitrary distribution of vorticity was found by Fraenkel [12] and Saffman [13] (see also Ref. [14]) as

$$U_0 = \frac{\Gamma}{4\pi R_0}\left\{\log\left(\frac{8R_0}{\sigma}\right) + A - \frac{1}{2}\right\}, \tag{2}$$

* Corresponding author. Tel.: +81 92 642 2762; fax: +81 92 642 2779.
E-mail address: yasuhide@math.kyushu-u.ac.jp (Y. Fukumoto).

0167-2789/$ - see front matter
doi:10.1016/j.physd.2008.02.003

where Γ is the circulation and

$$A = \lim_{r\to\infty}\left\{\frac{4\pi^2}{\Gamma^2}\int_0^r r' v_0(r')^2 \mathrm{d}r' - \log\left(\frac{r}{\sigma}\right)\right\}, \tag{3}$$

with $v_0(r)$ being the local velocity of circulatory motion of the fluid around the toroidal center circle, as a function only of the local distance r from the circle. In the absence of viscosity, $v_0(r)$ and therefore the local vorticity field may be arbitrary functions of r.

Viscosity acts to diffuse vorticity, and the motion ceases to be steady. For a vortex ring with its toroidal vorticity $\zeta(r,t)$ 'δ-function' concentrated on the circle of radius R_0, at a virtual instant,

$$\zeta(r,0) = \Gamma\delta(\rho - R_0)\delta(z - Z) \quad \text{at } t = 0, \tag{4}$$

with $r^2 = (\rho - R_0)^2 + (z - Z)^2$, it suffices to substitute, into (3), the Oseen diffusing vortex

$$\zeta_0 = \frac{\Gamma}{4\pi\nu t}\mathrm{e}^{-r^2/4\nu t}, \qquad v_0 = \frac{\Gamma}{2\pi r}\left(1 - \mathrm{e}^{-r^2/4\nu t}\right), \tag{5}$$

where ν is the kinematic viscosity and t is the time measured from the instant at which the core is infinitely thin. With this form, (2) supplemented by (3) becomes

$$U_0 = \frac{\Gamma}{4\pi R_0}\left\{\log\left(\frac{8R_0}{2\sqrt{\nu t}}\right) - \frac{1}{2}(1 - \gamma + \log 2)\right\}, \tag{6}$$

where $\gamma = 0.57721566\cdots$ is Euler's constant. Comparison with the result of numerical simulation of the axisymmetric Navier–Stokes equations [15] illustrates that validity of Saffman's formula (6) is limited to very small times ($\nu t/R_0^2 \ll 1$) [16].

Vortex rings observed in nature are not necessarily thin. Kelvin's formula is an asymptotic solution to $O(\varepsilon)$ for vorticity linear in the distance from the symmetric axis. Dyson [17] accomplished its extension to $O(\varepsilon^3)$ [18]. For this distribution, evidence is available that Dyson's formula fits rather well with the speed of Hill's spherical vortex, the fat limit of Fraenkel–Norbury's family of vortex rings [19]. This unexpected agreement stimulates us to pursue a higher-order correction to (2).

The method of matched asymptotic expansions has been previously developed for a systematic treatment of motion of slender vortex tubes [14,20], and was extended to second order in ε [21]. Derivation of the correction to Fraenkel–Saffman's formula (2) requests us to enter into the third order. A flood of nonlinear terms of a higher order in the Navier–Stokes equations makes our mathematical handling out of control. It was shown that the radius of the circle of vorticity centroid grows linearly in time due to the action of vorticity [22], but reduction of the expression for the speed of a vortex ring remains yet to be attained. The method of Lamb–Saffman–Rott–Cantwell [23,13,24] provides an efficient means.

We show how topological ideas help to bring in a further simplification. It is well known that a stationary configuration of vorticity, embedded in an inviscid incompressible fluid, is realizable as an extremal of energy on an iso-vortical sheet [25–27]. An iso-vortical sheet comprises volume-preserving diffeomorphisms, or smooth maps of fluid particles, with vorticity frozen into the fluid. For a moving state, this conditional variational principle should be augmented by a constraint. Benjamin [28] put forward a variational principle that an axisymmetric vortex ring moving steadily in an inviscid incompressible fluid is realizable as the maximum state of the kinetic energy H on an iso-vortical sheet, subject to the constraint of constant hydrodynamic impulse

$$\boldsymbol{P} = \frac{1}{2}\iiint \boldsymbol{x}\times\boldsymbol{\omega}\,\mathrm{d}V. \tag{7}$$

When translated into three dimensions, Kelvin–Benjamin's principle reads

$$\delta H - \boldsymbol{U}\cdot\delta\boldsymbol{P} = 0, \tag{8}$$

where the velocity $\boldsymbol{U}$ of the region plays the role of the Lagrangian multipliers.

An iso-vortical sheet is infinite dimensional. A family of solutions of the Euler equations include several parameters. By posing some relations on these parameters, we can maintain the solutions on a single iso-vortical sheet, and, when restricted to this family, the dimension of an iso-vortical sheet is reduced to finite. Thus the traveling speed of a vortex ring may be calculable through (8). This is indeed the case for the first-order velocity formula as listed in the book [29]. The principle (8) has a wider applicability as exemplified by a vortex ring governed by the Gross–Pitaevskii equation [30]. In this paper, we adapt this variational principle to deduce the $O(\varepsilon^3)$ correction to the traveling speed. At large Reynolds numbers, the viscosity plays a secondary role only of selecting vorticity profile, and the inviscid formula is applicable to give the correction term to Saffman's formula (6).

We begin with the general variational formulation in three dimensions (Section 2). After a statement of asymptotic expansions of the flow field, the kinetic energy and the impulse (Section 3), we recall the outer and inner solutions [22] in Sections 4 and 5 respectively. Thereafter, we calculate, in Section 6, the energy and the impulse to $O(\varepsilon^2)$ and present, in Section 7, a recipe for implementing (8) to produce the $O(\varepsilon^3)$ correction to Fraenkel–Saffman's formula (2) and Saffman's formula (6) for the traveling speed of vortex rings. It is highly probable that a vortex ring obeying the Euler equations is a maximum-energy state [28,31]. The upper bound of energy, if available, guarantees the existence of this extremal, and is furnished by a topological invariant [32]. Appendix gives a concise description for viewing this invariant as a variant of the helicity [33].

2. Variational principle

Roberts [34] proved the above principle for an axisymmetric vortex ring steadily translating in an inviscid fluid. Below, we extend this principle to three dimensions to gain an insight into the variational structure.