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Acoustic streaming flows in a cavity: An illustration of small-scale inviscid flow

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Abstract

Low Reynolds number flows are typically described by the equations of creeping motion, where viscous forces are dominant. We illustrate using particle image velocimetry (PIV) an example of small-scale boundary driven cavity flows, where forcing relies on viscous mechanisms at the boundary but resulting steady flow patterns are inviscid. Namely, we have investigated acoustic streaming flows inside an elastic spherical cavity. Here, the inviscid equations of fluid motion are not used as an approximation, but rather velocity fields independent of viscosity come as a result from the general solution of the creeping motion equations solved in the region interior to a sphere.

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1. Introduction

Low Reynolds number flows characterize flow phenomena where fluid velocities are very slow, viscosities are high or alternatively length scales of the flow are very small (e.g. microfluidics [1]), such that inertial forces are small compared to viscous forces. For incompressible Newtonian fluids, in the limit of vanishing Reynolds numbers, where $Re = \rho U L/\mu \ll 1$ (U is a characteristic fluid speed, L a characteristic length scale; μ and ρ the fluid's dynamic viscosity and density, respectively), the equations of creeping motion reduce to [2]

$$\nabla p - \mu \nabla^2 \underline{u} = 0, \tag{1}$$

$$\nabla \cdot \underline{u} = 0, \tag{2}$$

where inertial and transient terms may be neglected and the above equations describe, respectively, the conservation of momentum and mass (i.e. Stokes flow). Here, $\underline{u}$ is the velocity field and p the pressure. In three-dimensional (3D) flows, the component velocities, $\underline{u} = (u, v, w)^T$, may be related to a scalar stream function ψ [3,4]:

$$\underline{u} = \nabla\psi \times \underline{n} = \nabla \times \underline{\Psi}, \tag{3}$$

where $\underline{\Psi} = \psi \underline{n}$ is the stream function vector and $\underline{n}$ the unit normal vector perpendicular to the plane of $\nabla\psi$ and $\underline{u}$. For planar two-dimensional (2D) flows, Eq. (3) reduces effectively to a single stream function ψ where $u = \partial\psi/\partial y$ and $v = -\partial\psi/\partial x$. Under creeping motion, it is readily shown that $\underline{\Psi}$ satisfies the biharmonic equation [5]:

$$\nabla^4 \underline{\Psi} = 0, \tag{4}$$

where $\nabla^4 \underline{\Psi} = -\nabla^2 \underline{\omega}$, and $\underline{\omega} = \nabla \times \underline{u} = -\nabla^2 \underline{\Psi}$ defines the vorticity vector field. Neither the fourth-order differential equation for $\underline{\Psi}$ nor its boundary conditions, which govern the spatial distribution of $\underline{\Psi}$, contain Re such that streamlines are independent of viscosity μ [6].

Classic examples of such low Reynolds number flow phenomena are cavity flows which may illustrate slow internal recirculation induced by the translation of one or more of the containing walls [7,8], or driven by a shear flow over the cavity [9,10]. Here, we illustrate using flow visualization techniques (i.e. PIV), an original example of such low Reynolds number cavity flows, where forcing relies on viscous mechanisms at a solid–fluid interface, but resulting flow patterns are steady and inviscid. Namely, we have investigated acoustic streaming flows generated inside thin elastic spherical cavities. We demonstrate analytically that the resulting velocity fields are independent of viscosity as they may be captured by spherical harmonic functions which arise from the general

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solution of the creeping motion Eqs. (1) and (2) in spherical coordinates.

2. Solution to creeping motion inside a sphere

Following Lamb [11] and Happel and Brenner [2], the general solution to Eqs. (1) and (2) in spherical coordinates (r, θ, ϕ) may be given in terms of the velocity field:

$$\underline{u} = \sum_{n=1}^{\infty} \left[\nabla \times (\underline{r}\chi_n) + \nabla \Phi_n + \frac{(n+3)\,|r|^2\, \nabla p_n - n\underline{r}\,p_n}{2\mu(n+1)(2n+3)} \right], \quad (5)$$

with the assumption of finite velocities at the origin ($r = 0$), and where the scalar functions χ_n, Φ_n, and p_n are solid spherical harmonics defined as

$$\begin{aligned} \chi_n &= \frac{1}{n(n+1)} \left(\frac{r}{a}\right)^n Z_n^m(a,\theta,\phi), \\ \Phi_n &= \frac{a}{2n}\left(\frac{r}{a}\right)^n \left[(n+1)X_n^m(a,\theta,\phi) - Y_n^m(a,\theta,\phi)\right], \\ p_n &= \frac{\mu(2n+3)}{na}\left(\frac{r}{a}\right)^n \\ &\quad \times \left[Y_n^m(a,\theta,\phi) - (n-1)X_n^m(a,\theta,\phi)\right]. \end{aligned} \quad (6)$$

The above formulation makes use of the surface harmonics $Z_n^m(r,\theta,\phi)$, $Y_n^m(r,\theta,\phi)$, and $X_n^m(r,\theta,\phi)$ of degree m and order n. Each harmonic function takes the form $r^n P_n^m(\cos\theta)\,\mathrm{e}^{\mathrm{i}m\phi}$, where $P_n^m()$ are the associated Legendre functions.

The unknown scalar functions χ_n, Φ_n, and p_n are determined by matching the appropriate velocity and vorticity boundary conditions at the surface of the sphere ($r = a$). These conditions are described as follows:

$$\begin{aligned} u_n &= \frac{1}{|r|}\left(\underline{r}\cdot\underline{u}_s\right)\Big|_{r=a} = \sum_{n=1}^{\infty} X_n^m(a,\theta,\phi), \\ -|r|\left(\nabla\cdot\underline{u}_s\right)\big|_{r=a} &= \sum_{n=1}^{\infty} Y_n^m(a,\theta,\phi), \\ |r|\,\underline{\omega}_n = \underline{r}\cdot\left(\nabla\times\underline{u}_s\right)\big|_{r=a} &= \sum_{n=1}^{\infty} Z_n^m(a,\theta,\phi), \end{aligned} \quad (7)$$

where the velocity vector $\underline{u}_s(\theta,\phi)$ describes the surface velocity field at $r = a$ and ω_n is the vorticity component normal to the surface. For the problem at hand, one may use the fact that there is no normal velocity component, $u_n = 0$, at the surface of the sphere, such that

$$X_n^m(a,\theta,\phi) \equiv 0. \quad (8)$$

As a consequence, the full velocity field, $\underline{u}(r,\theta,\phi)$, in the region interior to the sphere can be described in terms of its surface normal vorticity, ω_n, and any contribution of source/sink distributions on the surface of the sphere resulting from $\nabla\cdot\underline{u}_s \neq 0$. By definition, $\underline{u}(r,\theta,\phi)$ is a solution of the biharmonic Eq. (4) and furthermore, it follows from Eqs. (5) and (6) that $\underline{u}(r,\theta,\phi)$ is independent of viscosity μ.

3. Acoustic streaming inside a thin elastic cavity

The propagation of sound waves in a fluid may lead to a bulk non-periodic motion of the fluid. This nonlinear phenomenon is called acoustic streaming [12] and is directly related to the quadratic convective terms of the flow field. We have investigated acoustic streaming flows generated at a solid–fluid interface by a sound wave of angular frequency ω. Experimental measurements of the resulting flow fields are based on particle image velocimetry (PIV) conducted inside millimeter-sized thin elastic spherical cavities of characteristic diameter $D = 2a$. The streaming phenomenon relies on a thin viscous boundary layer (Stokes layer) of thickness $\delta = (2\mu/\rho\omega)^{1/2}$ at the solid wall, where the no-slip boundary condition applies, while a steady-state solution independent of viscosity μ arises in the bulk of the flow away from the wall ($\delta \ll D$). Conceptually, outside the boundary layer δ, the driving force behind acoustic streaming is absorbed into the background hydrostatic pressure p in the momentum Eq. (1) [13].

3.1. Experimental methods

The experimental apparatus consists of a test cell allowing for optical access, enclosing a thin silicone elastomer film (50 μm thickness, $\rho = 1260$ kg/m^3), a loudspeaker and an imaging system (Fig. 1). The bottom of the test cell is connected to a graduated syringe such that spherical cavities may be inflated by injecting air which distends the silicone membrane. Typical cavities are generated at a 6 mm circular orifice opening, by inflating ~1.5–2 ml of air, resulting in a spherical cap with a characteristic diameter of $D \sim 6.5$–7 mm.

Depending on the excitation frequency, $f = \omega/2\pi$, acoustic waves are generated using a piezoelectric loudspeaker (3–20 kHz) or an electrostatic transducer (20–50 kHz) mounted onto one of the test cell faces and connected to a signal generator which delivers a sinusoidal electrical waveform. The imaging system consists of a progressive scan CCD camera with 15 Hz image acquisition rate and a resolution of 1008 × 1008 pixels triggered under computer control. The CCD camera is fitted onto a microscope with a field of view of about 7 × 7 mm, resulting in a spatial resolution of ~6.9 μm. The laser sheet is generated by a 150 mW diode laser making use of a light sheet optic. Due to the lighting and flow conditions (driven flows are typically ≪1 mm/s for the acoustic output power range), a pulsed illumination is not required. Rather, consecutive images with an exposure time of 1/15 s are recorded in a horizontal plane cutting through the inflated membrane at approximately maximum diameter. Since the elastomer film is not perfectly transparent, scattering effects of the light sheet may be observed close to the wall, perhaps compromising slightly PIV results in close proximity to the membrane, while the bulk of the measurement plane remains unaffected.

The syringe barrel is filled with air, seeded with oil droplets. 2D vector displacements are obtained with a custom PIV algorithm based on cross-correlation pattern matching with sub-pixel interpolation [14]. Typical measurements consist in the acquisition of 100 consecutive frames and velocity vectors obtained from independent image pairs are then time averaged following an average correlation method [17].

3.2. Flow fields

For the range of frequencies investigated, several steady streaming flows were observed (Fig. 2), with $U \ll 1$ mm/s