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Where do inertial particles go in fluid flows?

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Abstract

We derive a general reduced-order equation for the asymptotic motion of finite-size particles in unsteady fluid flows. Our *inertial equation* is a small perturbation of passive fluid advection on a globally attracting slow manifold. Among other things, the inertial equation implies that particle clustering locations in two-dimensional steady flows can be described rigorously by the Q parameter, i.e., by one-half of the squared difference of the vorticity and the rate of strain. Use of the inertial equation also enables us to solve the numerically ill-posed problem of source inversion, i.e., locating initial positions from a current particle distribution. We illustrate these results on inertial particle motion in the Jung–Tél–Ziemniak model of vortex shedding behind a cylinder in crossflow.

Keywords: Inertial particles; Slow manifolds; Singular perturbation theory; Nonautonomous systems

1. Introduction

Finite-size or inertial particle dynamics in fluid flows can differ markedly from infinitesimal particle dynamics: both clustering and dispersion are well-documented phenomena in inertial particle motion, while they are absent in the incompressible motion of infinitesimal particles. As we show in this paper, these peculiar asymptotic features are governed by a lower-dimensional inertial equation which we determine explicitly.

Let $\mathbf{u}(\mathbf{x}, t)$ denote the velocity field of a two- or three-dimensional fluid flow of density ρ_f, with $\mathbf{x}$ referring to spatial locations and t denoting time. The fluid fills a compact (possibly time-varying) spatial region $\mathcal{D}$ with boundary $\partial\mathcal{D}$; we assume that $\mathcal{D}$ is a uniformly bounded smooth manifold for all times. We also assume $\mathbf{u}(\mathbf{x}, t)$ to be r times continuously differentiable in its arguments for some integer $r \geq 1$. We denote the material derivative of $\mathbf{u}$ by

$$\frac{D\mathbf{u}}{Dt} = \mathbf{u}_t + (\nabla\mathbf{u})\mathbf{u},$$

where ∇ denotes the gradient operator with respect to $\mathbf{x}$.

Let $\mathbf{x}(t)$ denote the path of a finite-size particle of density ρ_p immersed in the fluid. If the particle is spherical, its velocity $\mathbf{v}(t) = \dot{\mathbf{x}}(t)$ satisfies the equation of motion (cf. Maxey and Riley [13] and Babiano et al. [2])

$$\begin{aligned}\rho_p\dot{\mathbf{v}} &= \rho_f\frac{D\mathbf{u}}{Dt}\\ &+(\rho_p-\rho_f)\mathbf{g}\\ &-\frac{9\nu\rho_f}{2a^2}\left(\mathbf{v}-\mathbf{u}-\frac{a^2}{6}\Delta\mathbf{u}\right)\\ &-\frac{\rho_f}{2}\left[\dot{\mathbf{v}}-\frac{D}{Dt}\left(\mathbf{u}+\frac{a^2}{10}\Delta\mathbf{u}\right)\right]\\ &-\frac{9\rho_f}{2a}\sqrt{\frac{\nu}{\pi}}\int_0^t\frac{1}{\sqrt{t-s}}\left[\dot{\mathbf{v}}(s)-\frac{\mathrm{d}}{\mathrm{d}s}\left(\mathbf{u}+\frac{a^2}{6}\Delta\mathbf{u}\right)_{\mathbf{x}=\mathbf{x}(s)}\right]\mathrm{d}s.\end{aligned} \quad (1)$$

Here ρ_p and ρ_f denote the particle and fluid densities, respectively, a is the radius of the particle, $\mathbf{g}$ is the constant vector of gravity, and ν is the kinematic viscosity of the fluid. The individual force terms listed in separate lines on the right-hand side of (2) have the following physical meaning: (1) force exerted on the particle by the undisturbed flow, (2) buoyancy force, (3) Stokes drag, (4) added mass term resulting

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from part of the fluid moving with the particle, and (5) the Basset–Boussinesq memory term. The terms involving $a^2 \Delta \mathbf{u}$ are usually referred to as the Fauxén corrections.

For simplicity, we assume that the particle is very small ($a \ll 1$), in which case the Fauxén corrections are negligible. We note that the coefficient of the Basset–Boussinesq memory term is equal to the coefficient of the Stokes drag term times $a/\sqrt{\pi \nu}$. Therefore, assuming that $a/\sqrt{\nu}$ is also very small, we neglect the last term in (2), following common practice in the related literature (Michaelides [14]). We finally rescale space, time, and velocity by a characteristic length scale L, characteristic time scale $T = L/U$ and characteristic velocity U, respectively, to obtain the simplified equations of motion

$$\dot{\mathbf{v}} - \frac{3R}{2}\frac{D\mathbf{u}}{Dt} = -\mu\,(\mathbf{v}-\mathbf{u}) + \left(1 - \frac{3R}{2}\right)\mathbf{g}, \tag{2}$$

with

$$R = \frac{2\rho_f}{\rho_f + 2\rho_p}, \qquad \mu = \frac{R}{St}, \qquad St = \frac{2}{9}\left(\frac{a}{L}\right)^2 Re,$$

and with t, $\mathbf{v}$, $\mathbf{u}$ and $\mathbf{g}$ now denoting nondimensional variables. Variants of Eq. (2) have been studied by Babiano, Cartwright, Piro and Provenzale [2], Benczik, Toroczkai and Tél [5], and Vilela, de Moura and Grebogi [20].

In Eq. (2), St denotes the particle Stokes number and $Re = UL/\nu$ is the Reynolds number. The density ratio R distinguishes neutrally buoyant particles ($R = 2/3$) from aerosols ($0 < R < 2/3$) and bubbles ($2/3 < R < 2$). In the limit of infinitely heavy particles ($R = 0$), Eq. (2) become the Maxey–Riley equations derived originally in [13]. The $3R/2$ coefficient represents the added mass effect: an inertial particle brings into motion a certain amount of fluid that is proportional to half of its mass. For neutrally buoyant particles, the equation of motion is simply $\frac{D}{Dt}(\mathbf{v}-\mathbf{u}) = -\mu\,(\mathbf{v}-\mathbf{u})$, i.e., the relative acceleration of the particle is equal to the Stokes drag acting on the particle.

Rubin, Jones and Maxey [17] studied (2) with $R = 0$ in the special case when $\mathbf{u}$ describes a two-dimensional cellular steady flow model. They used a geometric singular perturbation approach developed by Fenichel [8] to understand particle settling in the flow. The same technique was employed by Burns et al. [7] in the study of particle focusing in the wake of a two-dimensional bluff body flow, which is steady in a frame co-moving with the von Kármán vortex street. Recently, Mograbi and Bar-Ziv [15] discussed this approach for general steady velocity fields and made observations about the possible asymptotic behaviors in two dimensions.

Here we consider finite-size particle motion in general unsteady velocity fields, extending Fenichel's geometric approach from time-independent to time-dependent vector fields. Such an extension has apparently not been considered before in dynamical systems theory, thus the present work should be of interest in other applications of singular perturbation theory where the governing equations are non-autonomous. We construct an attracting slow manifold that governs the asymptotic behavior of particles in system (2). We also obtain an explicit dissipative equation, the *inertial equation*, that describes the flow on the slow manifold. This equation has half the dimension of the Maxey–Riley equation; this fact simplifies both the qualitative analysis of inertial dynamics and the numerical tracking of finite-size particles.

For two-dimensional steady flows, we use the inertial equation to give a complete description of the asymptotic behavior of aerosols, bubbles, and neutrally buoyant particles. For general unsteady flows, we show how the inertial equation can be used to locate the initial positions of dispersed particles. Such *source inversion* is not possible using the full Maxey–Riley equation, because for $\mu \gg 1$, the $-\mu\mathbf{u}$ term in (2) causes numerical solutions to blow up quickly in backward time. We illustrate the forward- and backward-time use of the inertial equation on the von Kármán vortex-street model of Jung, Tél and Ziemniak [12].

2. Singular perturbation formulation

The derivation of the equation of motion (2) is only correct under the assumption $\mu \gg 1$, which motivates us to introduce the small parameter

$$\epsilon = \frac{1}{\mu} \ll 1,$$

and rewrite (2) as a first-order system of differential equations:

$$\dot{\mathbf{x}} = \mathbf{v},$$
$$\epsilon\dot{\mathbf{v}} = \mathbf{u}(\mathbf{x},t) - \mathbf{v} + \epsilon\frac{3R}{2}\frac{D\mathbf{u}(\mathbf{x},t)}{Dt} + \epsilon\left(1 - \frac{3R}{2}\right)\mathbf{g}. \tag{3}$$

This formulation shows that $\mathbf{x}$ is a slow variable changing at $\mathcal{O}(1)$ speeds, while the fast variable $\mathbf{v}$ varies at speeds of $\mathcal{O}(1/\epsilon)$.

To transform the above singular perturbation problem to a regular perturbation problem, we select an arbitrary initial time t_0 and introduce the fast time τ by letting

$$\epsilon\tau = t - t_0.$$

This type of rescaling is standard in singular perturbation theory with $t_0 = 0$. The new feature here is the introduction of a nonzero present time t_0 about which we introduce the new fast time τ. This trick enables us to extend the existing singular perturbation techniques to unsteady flows.

Denoting differentiation with respect to τ by prime, we rewrite (3) as

$$\mathbf{x}' = \epsilon\mathbf{v},$$
$$\phi' = \epsilon,$$
$$\mathbf{v}' = \mathbf{u}(\mathbf{x},\phi) - \mathbf{v} + \epsilon\frac{3R}{2}\frac{D\mathbf{u}(\mathbf{x},\phi)}{Dt} + \epsilon\left(1 - \frac{3R}{2}\right)\mathbf{g}, \tag{4}$$

where $\phi \equiv t_0 + \epsilon\tau$ is a dummy variable that renders the above system of differential equations autonomous in the variables $(\mathbf{x},\phi,\mathbf{v}) \in \mathcal{D} \times \mathbb{R} \times \mathbb{R}^n$; here n is the dimension of the domain of definition $\mathcal{D}$ of the fluid flow ($n = 2$ for planar flows, and $n = 3$ for three-dimensional flows).