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Discretising systematically the Painlevé equations

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ABSTRACT

We apply a method that we have already proposed for the systematic discretisation of differential systems to the construction of discrete analogues of the Painlevé equations. The important point is that in the case of the Painlevé equations one needs to preserve the integrability. We show that this is indeed possible using our systematic procedure applied to the equations obtained from the Hamiltonian description of the Painlevé equations.

1. Introduction

Constructing discrete analogues of differential equations is not a particularly hard task. After all, when one uses some numerical algorithm for the integration of a given differential system one necessarily resorts to some discretisation, sometimes hidden inside the black-box integrator. However, as we have repeatedly stressed, this is not the best approach. While a ready-made numerical routine may give a reasonably accurate solution, we believe that an optimal integrator should preserve as many of the properties of the continuous system as possible. The difficulty one faces in this approach is how to obtain these tailor-made discretisations. In a series of papers [1–3] we have constructed discrete analogues of a host of nonlinear dynamical systems. While our intent was explicitly stated in all these works, the method for obtaining the adequate discretisations was largely intuitive and, in a sense, not always easy to explain. This last point was recently remedied with the introduction of a systematic discretisation method [4] which allowed us to recover and interpret the discretisations previously proposed.

Painlevé equations [5] pose a greater challenge with respect to discretisation. Here the main property to preserve is integrability. This is the crucial one since all the nice properties of the Painlevé equations proceed from the integrable character. We must hasten to say that the problem of constructing integrable discrete analogues of the Painlevé equations has been solved for quite some time now. As a matter of fact, as early as 1992 a systematic construction of discrete Painlevé equations was proposed [6]. This approach was based on the existence of an integrability criterion [7] which allowed the derivation of the proper, i.e. integrable, deautonomisations of the QRT [8] mapping. Since the solution of the latter is given in terms of elliptic functions, the approach made sense. Difficulties did, of course, exist. For instance, it was not initially possible to obtain an integrable discrete analogue of P_{VI}: this was realised much later [8], when the accumulation of results on discrete Painlevé equations made possible a different approach.

The problem we are going to address in this paper is whether one can, using the systematic discretisation procedure we introduced in [4], discretise the Painlevé equations while preserving integrability. The difference with all the previous approaches is that here we are going to apply a discretisation procedure to the continuous forms. The method developed in [4] was tailored to first-order systems. Since we are in principle dealing with second-order equations we shall begin by extending our approach to second-order systems.

2. A brief summary of the discretisation approach and an extension to second-order systems

Before proceeding to the discretisation of Painlevé equations we present a summary of the systematic discretisation method introduced in [4]. In all that follows we shall restrict ourselves to the autonomous case where the coefficients of the equations do not depend on the independent variable: once an integrable autonomous mapping is obtained, its integrable deautonomisation is usually quite straightforward.

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Our discretisation procedure is based on an *ad hoc* linearisation of the differential system and a subsequent discretisation combined with a Padé-type approximation of the exponential operator. This can be better understood through an example. We start from a first-order equation

$$x' = \alpha x + \beta \tag{2.1}$$

the solution of which is

$$x(t) = ce^{\alpha t} - \frac{\beta}{\alpha}. \tag{2.2}$$

A time-discretisation is introduced with a step ϵ. Computing $x(t+\epsilon)$ we find

$$x(t+\epsilon) = ce^{\alpha(t+\epsilon)} - \frac{\beta}{\alpha} = e^{\alpha\epsilon}\left(x(t) + \frac{\beta}{\alpha}\right) - \frac{\beta}{\alpha}. \tag{2.3}$$

At this stage we introduce a rational approximation for the exponential

$$e^{\sigma} = \frac{1+(\lambda+1)\sigma}{1+\lambda\sigma} \tag{2.4}$$

which has some freedom since the parameter λ is not fixed a priori. Finally, as is customary, we indicate the discretised time by an index which leads to

$$x_{n+1} = \frac{1+(\lambda+1)\alpha\epsilon}{1+\lambda\alpha\epsilon}x_n + \frac{\epsilon\beta}{1+\lambda\alpha\epsilon}. \tag{2.5}$$

In [4] the discretisation based on (2.5) was applied to several nonlinear systems where the nonlinearity was of the "bilinear" type, meaning that the α appearing in (2.1) could be a linear combination of the remaining variables of the system. However nonlinearities involving squares (or even higher powers) may well exist and thus it was natural to extend the method in order to deal with this type of nonlinearity. The simplest case with higher nonlinearities is that of the Riccati equation. Introducing (for obvious reasons) a factorisation of the terms we start from

$$x' = (\alpha x + 2\beta)x + \gamma. \tag{2.6}$$

The application of the discretisation (2.5) is straightforward. We obtain

$$x_{n+1} = \frac{(1+(\lambda+1)\epsilon(\alpha x_n + 2\beta))x_n + \epsilon\gamma}{1+\lambda\epsilon(\alpha x_n + 2\beta)}. \tag{2.7}$$

For a generic value of λ, this is not an acceptable discretisation since it does not allow for reverse evolution, i.e. towards diminishing values of n. However by taking $\lambda = -1$ we find the mapping

$$x_{n+1} = \frac{x_n + \epsilon\gamma}{1 - \epsilon\alpha x_n - 2\epsilon\beta} \tag{2.8}$$

which is of a homographic form, which constitutes an integrable discretisation of the Riccati equation. This is not the only possible form of the discrete Riccati. As explained in [4] another possibility does exist

$$x_{n+1} = \frac{(1+\epsilon\beta)x_n + \epsilon\gamma}{1 - \alpha\epsilon x_n - \epsilon\beta}. \tag{2.9}$$

The derivation of (2.9) presented in [4] was based on the solutions of (2.6). Here we can give a different derivation which could be used as an intuitive rule for further discretisations. The mapping in (2.9) can be obtained simply from (2.6) if one chooses the following ansatz $x' \to (x_{n+1} - x_n)/\epsilon$, $x^2 \to x_{n+1}x_n$ and for the linear term $x \to (x_{n+1} + x_n)/2$.

While the discretisation of first-order systems may well be sufficient, since one can always rewrite a higher order differential equation as a system of first-order ones, it is interesting, in view of the application to Painlevé equations, to extend our method to second-order systems. We start with a simple linear second-order equation which does not contain a first-derivative term:

$$x'' = \alpha^2 x + \beta. \tag{2.10}$$

The solution of (2.10) is

$$x(t) = ce^{\alpha t} + de^{-\alpha t} - \frac{\beta}{\alpha^2}. \tag{2.11}$$

As previously we introduce a discretisation step ϵ and find that the combination $x(t+\epsilon) + x(t-\epsilon)$ can be expressed in terms of x alone:

$$x_{n+1} + x_{n-1} = (e^{\alpha\epsilon} + e^{-\alpha\epsilon})x_n + \frac{e^{\alpha\epsilon} + e^{-\alpha\epsilon} - 2}{\alpha^2}\beta \tag{2.12}$$

where we have used the index n in order to indicate the discretised time, i.e. $t = n\epsilon$. The rational approximation to the exponential (2.4) is next introduced leading to

$$x_{n+1} + x_{n-1} = \frac{2(1-\lambda(\lambda+1)\alpha^2\epsilon^2)x_n - 2\lambda\epsilon^2\beta}{1-\lambda^2\alpha^2\epsilon^2} \tag{2.13}$$

which constitutes the discretisation of (2.10).

3. Discrete analogues of Painlevé equations I and II

Turning to the discretisation of the Painlevé equations we shall repeat the clarification that we shall deal only with autonomous forms without attempting a discretisation of the independent variable which appears explicitly in the continuous equation. We shall treat this variable as a mere parameter. The integrable deautonomisation of the mappings obtained with our procedure does not present any particular difficulty.

Our starting point is the continuous Painlevé I equation

$$x'' = x^2 + z \tag{3.1}$$

where the derivative is taken with respect to z. As in the case of the Riccati equation we "factorise" the x^2 term and consider that x plays the role of the α^2 in Eq. (2.10). A straightforward application of the discretisation (2.13) leads then to

$$x_{n+1} + x_{n-1} = \frac{2(1-\lambda(\lambda+1)x_n)x_n - 2\lambda z}{1-\lambda^2 x_n} \tag{3.2}$$

where ϵ has been taken to 1. Eq. (3.2) is not integrable for all values of λ. Still, two specific values which lead to integrable mapping do exist. First taking $\lambda = -1$ we find

$$x_{n+1} + x_{n-1} = \frac{2x_n + 2z}{1 - x_n} \tag{3.3}$$

and translating $x = -4y + 1$ we obtain

$$y_{n+1} + y_{n-1} = 1 + \frac{a}{y_n} \tag{3.4}$$

where $a = (-1-z)/8$. Eq. (3.4) has been identified in [9] as a discrete analogue of $P_{\rm I}$ (to be precise, when a is a linear function of the independent variable n). However another value of λ does exist. We start by adding x_n to both members of (3.2) and choose λ so as to make the term proportional to x^2 disappear from the rhs of the equation. We find the value $\lambda = -2/3$ and by introducing the variable y through $x = -27y/2 + 9/4$ we obtain

$$y_{n+1} + y_{n-1} + y_n = 1 + \frac{a}{y_n} \tag{3.5}$$

where $a = -1/12 - 4z/243$. Eq. (3.5) is (again when a is a linear function of n) the "standard" form of the discrete Painlevé I.