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A canonical model for gradient frequency neural networks

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ABSTRACT

We derive a canonical model for *gradient frequency neural networks* (GFNNs) capable of processing time-varying external stimuli. First, we employ normal form theory to derive a fully expanded model of neural oscillation. Next, we generalize from the single oscillator model to heterogeneous frequency networks with an external input. Finally, we define the GFNN and illustrate nonlinear time-frequency transformation of a time-varying external stimulus. This model facilitates the study of nonlinear time-frequency transformation, a topic of critical importance in auditory signal processing.

1. Introduction

Most existing work on neural oscillator networks focuses on the intrinsic dynamics of networks with a homogeneous distribution of oscillator frequencies. The truncated normal form (see, e.g., [1–3]) provides a suitable canonical model for the study of network dynamics in such cases (e.g., [4,5]) because it includes all resonant terms necessary to understand the interactions of oscillators with equal (or ϵ-close) frequencies. We wish to study heterogeneous frequency oscillator networks that process external stimuli because this topic is of critical relevance to understanding auditory processing. A growing body of evidence suggests that the auditory nervous system is highly nonlinear, and that nonlinear transformations of auditory stimuli have important functional consequences [6–10]. Thus, findings and interpretations about the dynamics of heterogeneous networks may have relevance for cochlear modeling [11–14] and brainstem physiology [9,15,10], as well as pitch and music perception [16–18].

Our goal is to develop a model of neural oscillation that facilitates investigations of the nonlinearities that underlie auditory physiology and perception. Our approach involves specifying an appropriate class of oscillators and transforming it to a generic form, known as a *normal form*, via *normal form theory* [3,19,20,2,1]. Normal forms are important analytical tools in the local analysis of dynamical systems in the neighborhood of elementary solutions such as equilibria and periodic orbits. The principal goal of normal form theory is to obtain local coordinates in terms of which a dynamical system near an elementary solution has a “simplest” form or canonical representation which, in turn, can facilitate its analysis. The structures of the normal forms we consider are in terms of “resonances” (e.g., [19,20,1]).

An important issue that arises when considering the external stimulation of an oscillator network is that the structure of the input to any given oscillator is not known in advance. Moreover, at any given time, the stimulus may contain a combination of external and internal (within the network) signals. The key to our approach to obtaining a canonical model is to fully expand the nonlinearities and the resonant terms of the normal form for each oscillator based on its *natural frequency*. Any frequencies in the stimulus that “resonate” with the natural frequency will have significant effects on the canonical oscillator’s dynamics. This approach leads us to consider external stimulation at the level of the canonical model instead of at the level of the original class of oscillators, simplifying the analytical nature of the resulting model. In what follows, we define *gradient-frequency neural networks* (GFNN’s) and derive a canonical GFNN. We compare the nonlinear time-frequency transformation of an acoustic stimulus by a GFNN based on Wilson–Cowan oscillators and a GFNN based on our canonical model.

2. A truncated canonical model for neural oscillator networks

Consider the general system of coupled neural oscillators modeled by the network equations:

$$\begin{aligned} \dot{u}_i &= f_i(u_i, v_i, \lambda) + \epsilon p_i(u_1, v_1, \ldots, u_n, v_n, \epsilon) \\ \dot{v}_i &= g_i(u_i, v_i, \lambda) + \epsilon q_i(u_1, v_1, \ldots, u_n, v_n, \epsilon). \end{aligned} \tag{1}$$

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In Eq. (1), $\{u_i, v_i\} \subset \mathbb{R}$ represent the coordinates of the state of the ith oscillator. λ represents the set of parameters of the functions f_i and g_i. $\epsilon > 0$ is a connectivity parameter.

Appendix A, briefly reviews one of the standard procedures for obtaining normal forms and clarifies the relationship between a normal form and a canonical model. As discussed in Appendix A, the classical analysis leading to a normal form for Eq. (1) involves a coordinate transformation, dependent on the Jacobian matrix of the system, and an expansion of the nonlinearities. For the class of neural oscillators represented by Eq. (1), normal form theory (see [1,5,2,21,3,22]) leads to a generic form (Eq. (2)) in a new complex valued state variable, z, resulting from the coordinate transformation.

$$\dot{z}_i = z_i(a_i + b_i|z_i|^2) + x_i(t) + h.o.t., \quad i \in \{1, \ldots, n \in \mathbb{Z}^+\}$$

where

$$x_i(t) = \sum_{j \neq i}^{n} c_{ij} z_j, \quad \{a_i, b_i, c_{ij}, z_i\} \subset \mathbb{C}. \tag{2}$$

Eq. (2) is also a canonical model representing the local dynamics about an Andronov–Hopf bifurcation for the entire class of neural oscillators given by Eq. (1). It has complex-valued parameters a_i and b_i which can be related, via the coordinate transformation (see [1,5]), to the parameters of the original system (Eq. (1)). In standard complex form, $a_i = \alpha_i + \imath\omega_i$, where ω_i is the *natural frequency* or *eigenfrequency* of the ith oscillator, and $b_i = \beta_i + \imath\delta_i$. The complex coefficients c_{ij} represent the coupling strengths among the oscillators. Note that $x_i(t)$ represents the total combination of input to the ith oscillator, including all coupled inputs from other oscillators. The system given by Eq. (2) is an appropriate model for the study of a system such as Eq. (1) near one of its bifurcation points, e.g., at an Andronov–Hopf bifurcation, where each oscillator will have a specific frequency. Normal form models, with the addition of an external stimulus, i.e., $x_i(t) = s(t) + c_{ij}z_j$, have been proposed to capture some functionally important nonlinearities of the mammalian cochlea [23,13].

Eq. (2) is referred to as a *truncated normal form* because the expansion of the nonlinearities (Eq. (1)) is truncated, effectively ignoring the higher order terms, $h.o.t.$ It is important to realize, however, that any interactions between oscillators of different frequencies in Eq. (1) would be captured in the higher order terms of Eq. (2). But if it is assumed that all oscillators in the network have frequencies that are ϵ-close (see, e.g., [1] Thm. 5.8 Pg. 165), then the higher order terms have a negligible effect on the dynamics of the system, and there is a canonical model given by Eq. (3).

$$\dot{z}_i = z_i(a_i + b_i|z_i|^2) + \sum_{j \neq i}^{n} c_{ij} z_j + \mathcal{O}(\sqrt{\epsilon}), \quad i \in \{1, \ldots, n \in \mathbb{Z}^+\}. \tag{3}$$

The behavior of the canonical system Eq. (3) can be further understood by transforming it to polar coordinates (Eq. (4)) by expressing z_i in terms of its amplitude r_i and phase ϕ_i: $z_i(t) = r_i(t)e^{\imath\phi_i(t)}$. The coupled input $x_i(t) = \sum_{j \neq i}^{n} c_{ij} z_j$ can be represented in polar form as well, say, by $F_i(t)e^{\imath\theta_i(t)}$ where F_i and θ_i represent the amplitude and phase, respectively. This polar representation of the model allows for the independent study of amplitude and phase dynamics, and makes the meaning of the parameters explicit.

$$\dot{r}_i = r_i(\alpha_i + \beta_i r_i^2) + F_i \cos(\phi_i - \theta_i) + \mathcal{O}(\sqrt{\epsilon})$$
$$\dot{\phi}_i = \omega_i + \delta_i r_i^2 - \frac{F_i}{r_i} \sin(\phi_i - \theta_i) + \mathcal{O}(\sqrt{\epsilon}). \tag{4}$$

2.1. Neural oscillator network with input

Because of their theoretical and practical importance, we want to study nonlinear oscillator networks under the influence of complex acoustic stimuli. When external input $(\rho_{u_i}(t), \rho_{v_i}(t)) \in \mathbb{R}^2$ is specified in the original system as shown in Eq. (5) then the transformative procedure employed to obtain the normal form also transforms the external input.

$$\dot{u}_i = f_i(u_i, v_i, \lambda) + \epsilon p_i(u_1, v_1, \ldots, u_n, v_n, \rho_{u_i}(t), \epsilon)$$
$$\dot{v}_i = g_i(u_i, v_i, \lambda) + \epsilon q_i(u_1, v_1, \ldots, u_n, v_n, \rho_{v_i}(t), \epsilon). \tag{5}$$

This transformation leads to significant complexities in deriving a canonical model. For example, the expressions representing coupling coefficients can involve limits of integrals that are not necessarily convergent, or other complex expressions ([1] Thm. 5.10 p. 176). Moreover, if the input is resonant with the oscillators' natural frequencies, the canonical model may be difficult or impossible to derive. Due to such complexities, known methods for deriving canonical models cannot be applied.

Here we consider a different approach, taking into account the fact that canonical models are generic models for a system's local dynamics about one of its attractors. In this paradigm, the canonical model for a system without external input is considered as the fundamental model representing the intrinsic dynamics of a system. This essentially models a system at one of its behavioral modes. The generic mode of the system and its resonant behavior to input is precisely the case we are interested in as it corresponds to important physical situations (e.g., [5,13]). Thus, Eq. (2) becomes the fundamental model of interest, and additive external input $s(t) \in \mathbb{C}$ to oscillator z_i can be included in the coupling term $x_i(t)$ as follows.

$$x_i(t) = s(t) + \sum_{j \neq i}^{n} c_{ij} z_j. \tag{6}$$

Next, we consider the case in which a network of neural oscillators can have different natural frequencies, perhaps spanning several orders of magnitude. In this case, intrinsic oscillator frequencies do not need to be ϵ-close. Such freedom makes the analysis of such systems more difficult, but the dynamics are more interesting in terms of new behaviors. We then consider an external input whose frequency content is not known a priori. We fully expand the nonlinearities and resonances contained in the higher order terms $h.o.t.$ of Eq. (2), to incorporate the responses to an input of unknown frequency. We then compare the response of the canonical model to the input with that of a particular neural oscillator model.

3. A fully expanded canonical model for a single neural oscillator with an input

In this section we derive a fully expanded canonical model corresponding to the dynamical system Eq. (1) by continuing the expansion of higher order terms ($h.o.t.$) of the normal form near an Andronov–Hopf bifurcation. Higher order terms of the normal form are necessary to capture the response of an oscillator to an input that is not close to its natural frequency. We employ the linear relationship, or *resonance*, given by Eq. (A.2) in terms of the system's eigenvalues. Note that near the Andronov–Hopf bifurcation, the canonical oscillator frequencies $\{\omega_1, \ldots, \omega_n\}$ are absolute values of the eigenvalues of the system represented by Eq. (1) (see [1,5]). In this case, the resonance relationship becomes:

$$\omega_{res} = n_1\omega_1 + \cdots + n_m\omega_n$$
$$\text{where } \{m, n\} \subset \mathbb{Z}^+, \quad res \in \{n_1, \ldots, n_m\} \subset \mathbb{Z}^+. \tag{7}$$

This relationship leads to *resonant monomials*, which correspond to resonances among the eigenvalues of the original system that cannot be eliminated from the normal form [1]. Resonant monomials capture harmonics, subharmonics, and higher order combinations of the input frequencies. For example, we can