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Effects of parametric disorder on a stationary bifurcation

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Abstract

Effects of a frozen random contribution to the control parameter are investigated in terms of the complex Ginzburg–Landau equation with real coefficients. The threshold of the bifurcation from the homogeneous basic state is reduced by a random contribution even with a vanishing spatial mean value, as shown by three different approaches, by a perturbation calculation, by a self-consistent iteration method and by a fully numerical solution of the linear part of the Ginzburg–Landau equation. For arbitrary random contributions the nonlinear stationary solutions are numerically determined and in the limit of small random amplitudes analytical expressions are derived in terms of two different perturbation expansions, which cover already several related trends beyond threshold. For instance, the spatial modulations of the solutions increase with the noise amplitude, but decrease with increasing distance from threshold.

Keywords: Pattern formation; Pattern selection; Hydrodynamic stability

1. Introduction

Pattern formation is investigated particularly in spatially uniform systems in order to understand at first the essence of the underlying basic mechanisms [1,2]. However, real systems are not always perfectly uniform and include heterogeneities, which may become nonnegligible in various situations. Heterogeneities modify a bifurcation scenario by shifting for instance the threshold and the nonlinear solution behavior. How strong are such changes and how robust are generic pattern formation processes with respect to small inhomogeneities? Above which strength do the inhomogeneities change the bifurcation behavior qualitatively? Here we focus on heterogeneity effects on a stationary bifurcation, where the spatially varying parameters occur multiplicatively in the respective model equations.

Rayleigh–Bénard convection [3,4] and Taylor–Vortex flow [3,5] are two systems where the effects of inhomogeneities have already been investigated rather early. In both systems the effects of ramps, of periodic modulations, or of statistical distributed heterogeneities on different aspects of pattern formation, such as wavelength selection etc., have been studied [6–25]. In several of these examples the spatial variation of the parameters was restricted to one direction [6–17] and in others either the modulation depends on two spatial variables or the effects of a one-dimensional modulation on a two-dimensional pattern was investigated [18–25]. Recently, a number of investigations about heterogeneity effects were also focusing on the Turing instability in chemical reactions [26–30], on excitable media [31–34], or on optical systems [35].

Here, we investigate the influence of a time-independent and spatially periodic or spatially varying random contribution to the control parameter on a supercritical bifurcation from a homogeneous basic state to a spatially periodic state. Close to the threshold of such a supercritical bifurcation with wave number q_c, a real field $u(x, t)$ may be written as a product of the fast varying function $\propto e^{iq_c x}$ and the amplitude $A(x, t)$

$$u(x, t) = A(x, t)e^{iq_c x} + A^*(x, t)e^{-iq_c x}. \qquad (1)$$

With increasing distance from the threshold higher harmonic contributions are also needed in order to describe $u(x, t)$. There are many systems showing a transition to a one-dimensional periodic state of this type, as for instance the famous Rayleigh–Bénard convection, the Taylor vortex flow, electroconvection [36] or the Turing instability. Here we will

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focus on values of the control parameter close to the threshold, where only long-wavelength modulations of the spatially periodic function $\propto \exp(\mathrm{i}q_c x)$ are relevant. Such variations are commonly described by a slowly varying amplitude $A(x,t)$ of the periodic pattern given in Eq. (1) for which the well-known Ginzburg–Landau equation [37,38]

$$\tau_0 \partial_t A = \varepsilon A + \xi_0^2 \partial_x^2 A - g|A|^2 A \tag{2}$$

may be derived. Here τ_0 is the relaxation time, ε measures the distance from threshold of the spatially periodic pattern, ξ_0 is the coherence length and the nonlinear coefficient g determines the amplitude of the pattern as a function of the control parameter ε, as for instance with $A \propto \sqrt{\varepsilon/g}$ and $g > 0$ close to a supercritical (forward) bifurcation. This amplitude equation is one of the simplest models describing a bifurcation from an initial state $A = 0$ to a stationary and spatially varying pattern $A \neq 0$. A spatial modulation of the control parameter $\varepsilon \to \varepsilon + M(x)$, which describes for instance major effects of a rough container boundary in Rayleigh–Bénard convection as estimated in Ref. [12], leads to a modification of the bifurcation as shown in this work. It leads for instance to a shift of the threshold, but leaves the bifurcation still perfect, whereas an additive contribution $f(x,t)$ in Eq. (2) makes the bifurcation imperfect.

In Section 2 we describe the model and three different types of modulations of the control parameter. The effects of these modulations on the threshold are calculated in Section 3 by solving the linear part of Eq. (3) by three different methods, whereby their strengths and the deficiencies are compared with each other. In Section 3.1 a perturbation method is employed, in Section 3.2 a full numerical solution in Fourier space is given and in Section 3.3 a self-consistent approach is introduced. Beyond the threshold the dependence of the nonlinear solution on the distance from the threshold is also changed by the modulation of the control parameter as described in Section 4. In Section 4.1 the nonlinear equation is solved numerically for a typical set of parameters and in Section 4.2 the nonlinear solutions are determined in the range of small modulation amplitudes by the Poincaré–Lindstedt expansion. The results of this expansion are compared with the results obtained by solving the nonlinear equation numerically. A modified expansion of the nonlinear solution far beyond threshold is described in Section 4.3 and the work ends with a discussion and concluding remarks in Section 5.

2. Model equation

The effects of a time-independent random contribution $M(x)$ to the control parameter are investigated near a bifurcation point or phase transition in terms of the complex Ginzburg–Landau equation with real coefficients

$$\tau_0 \partial_t A = \left[\varepsilon + M(x) + \xi_0^2 \partial_x^2\right] A - g|A|^2 A. \tag{3}$$

$M(x)$ is assumed to be either spatially periodic or a random function, as specified in the following subsection. In addition, the amplitude of $M(x)$ is assumed to be of the order of ε and the spatial variation of $M(x)$ is considered to be slow on the length scale $2\pi/q_c$.

2.1. Three types of spatial modulations $M(x)$

The effects of three different types of the modulation function $M(x)$ are investigated in this work. Studies of spatially periodic modulations of the control parameter already have some tradition [6–15] and some trends to be expected for a random function $M(x)$ in Eq. (3) can already be investigated in terms of a periodic function

$$M(x) = 2G\cos(kx). \tag{4}$$

For a randomly varying modulation $M(x) = \xi(x)$, we assume a vanishing mean value and a δ-correlated second moment [39]:

$$\langle \xi(x) \rangle = 0, \tag{5a}$$

$$\langle \xi(x)\xi(x') \rangle = D\,\delta(x - x'). \tag{5b}$$

The amplitude D in Eq. (5b) is a measure for the noise intensity, and the δ-correlation expresses that the random function $M(x)$ is statistically independent at each location $x \neq x'$. The Fourier transform of the correlation function (5b) is also δ-correlated $\langle \xi^*(q)\xi(q') \rangle = 2\pi D\delta(q - q')$ and otherwise independent of the wave number.

In a third example we assume an Ornstein–Uhlenbeck process $M(x) = \omega(x)$ in space, where $\omega(x)$ is generated by a white noise $\xi(x)$ via the first order differential equation [39,40]

$$\frac{\partial \omega(x)}{\partial x} = -\frac{\omega(x)}{\ell} + \frac{\xi(x)}{\ell} \tag{6}$$

for different values of the correlation length ℓ. $\omega(x)$ is a so-called colored noise with vanishing mean value, where $\omega(x)$ and $\omega(x')$ at different sites $x \neq x'$ are correlated with an exponential decay of the correlation on a typical length ℓ as follows:

$$\langle \omega(x) \rangle = 0, \tag{7a}$$

$$\langle \omega(x)\,\omega(x') \rangle = \frac{D}{2\,\ell}\,\mathrm{e}^{-|x-x'|/\ell}. \tag{7b}$$

The exponential decay in Eq. (7b) leads to a wave number dependence of its Fourier transformation $\hat{\omega}(q)$

$$\langle \hat{\omega}^*(q)\,\hat{\omega}(q') \rangle = \frac{2\pi D\delta(q - q')}{1 + \ell^2 q^2}. \tag{8}$$

The correlation function of white noise is recovered from Eqs. (7b) and (8) in the limit $\ell \to 0$ by keeping D fixed.

Numerically the model in Eq. (3) may be solved on a finite number of grid points N with a grid spacing $\Delta x = L/N$ and the position $x_j = j\Delta x$ of the sites. An alternative approach is a solution in terms of N Fourier modes, as presented in Section 2.2. On a discrete lattice the autocorrelation function of the random process $\xi(x)$ takes the form

$$\langle \xi(x_i)\xi(x_j) \rangle = \frac{D}{\Delta x}\,\delta_{i,j}, \tag{9}$$