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Physica D 217 (2006) 142–152

PHYSICA D

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Invariant dynamical systems embedded in the N-vortex problem on a sphere with pole vortices

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Received 28 October 2005; received in revised form 1 March 2006; accepted 3 April 2006
Available online 3 May 2006
Communicated by Y. Nishiura

Abstract

We are concerned with the system of N vortex points on a sphere with two fixed vortex points at the poles. This article gives a reduction method of the system to invariant dynamical systems when all the vortex points have the same strength. It is carried out by considering the invariant property of the system with respect to the shift and pole reversal transformations, for which the polygonal ring configuration of the N vortex points at the line of latitude, called the "N-ring", remains unchanged. We prove that there exists a $2p$-dimensional invariant dynamical system reduced by the p-shift transformation for an arbitrary factor p of N. The p-shift invariant system is equivalent to the p-vortex-points system generated by the averaged Hamiltonian with the modified pole vortices. It is also shown that the system can be reduced by the pole reversal transformation when the pole vortices are identical. Since the reduced dynamical systems are defined in the linear space spanned by the eigenvectors given in the linear stability analysis for the N-ring, we obtain the inclusion relation among the invariant reduced dynamical systems. This allows us to decompose the system of a large number of vortex points into a collection of invariant reduced subsystems.

Keywords: Vortex points; Flow on a sphere; Reduction method; Invariant dynamical systems

1. Introduction

We consider the motion of the inviscid and incompressible flow on a sphere. Specifically, we focus on the motion of the vortex points, in which the vorticity concentrates discretely. Since the strength of the vortex point, which is the circulation around the point, is conserved according to Kelvin's theorem, the vortex point behaves like a material point, and it is advected by the velocity field that the other vortex points induce.

Now, let (Θ_m, Ψ_m) denote the position of the mth vortex point in the spherical coordinates. The equations of the N vortex points with the identical strength $\Gamma^{(N)} = 2\pi/N$ on the sphere are given by

$$\dot{\Theta}_m = -\frac{\Gamma^{(N)}}{4\pi} \sum_{j\neq m}^{N} \frac{\sin\Theta_j \sin(\Psi_m - \Psi_j)}{1-\cos\gamma_{mj}} \equiv F_m, \tag{1}$$

$$\begin{aligned}\dot{\Psi}_m = &-\frac{\Gamma^{(N)}}{4\pi \sin\Theta_m} \\ &\times \sum_{j\neq m}^{N} \frac{\cos\Theta_m \sin\Theta_j \cos(\Psi_m-\Psi_j) - \sin\Theta_m \cos\Theta_j}{1-\cos\gamma_{mj}} \\ &+ \frac{\Gamma_1}{4\pi}\frac{1}{1-\cos\Theta_m} - \frac{\Gamma_2}{4\pi}\frac{1}{1+\cos\Theta_m} \equiv G_m, \\ &m = 1, 2, \ldots, N,\end{aligned} \tag{2}$$

in which γ_{mj} represents the central angle between the mth and the jth vortex points, and

$$\cos\gamma_{mj} = \cos\Theta_m \cos\Theta_j + \sin\Theta_m \sin\Theta_j \cos(\Psi_m - \Psi_j).$$

The last two forcing terms in the Eq. (2) represent the flow fields induced by the pole vortices. The strengths of the north and the south pole vortices are denoted by Γ_1 and Γ_2 respectively. They are formally introduced in order to incorporate an effect of rotation of the sphere locally.

The Eqs. (1) and (2) define the dynamical system in the $2N$-dimensional phase space $\mathbb{P}_N \equiv [0,\pi]^N \times (\mathbb{R}/2\pi\mathbb{Z})^N$, which

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0167-2789/$ - see front matter
doi:10.1016/j.physd.2006.04.002

we rewrite in the following vector form:

$$\frac{\mathrm{d}\vec{x}}{\mathrm{d}t} = \mathbb{F}(\vec{x}),$$

where the map $\mathbb{F} : \mathbb{P}_N \to \mathbb{R}^{2N}$ gives the vector field for the position $\vec{x} \in \mathbb{P}_N$,

$$\mathbb{F} : (\Theta_1, \ldots, \Theta_N, \Psi_1, \ldots, \Psi_N) \mapsto (F_1, \ldots, F_N, G_1, \ldots, G_N).$$

We call the dynamical system the "N-vortex system" or the "N-vortex problem" with the identical strength. This is the Hamiltonian dynamical system [7,12], whose Hamiltonian is represented by

$$\begin{aligned} H = &-\frac{\left(\Gamma^{(N)}\right)^2}{8\pi} \sum_{m=1}^{N} \sum_{j\neq m}^{N} \log(1-\cos\gamma_{mj}) \\ &- \frac{\Gamma_1 \Gamma^{(N)}}{4\pi} \sum_{m=1}^{N} \log(1-\cos\Theta_m) \\ &- \frac{\Gamma_2 \Gamma^{(N)}}{4\pi} \sum_{m=1}^{N} \log(1+\cos\Theta_m). \end{aligned} \tag{3}$$

The solution of the Eqs. (1) and (2) exists globally in time, since the self-similar collapse of the vortex points never occurs when the strengths of the vortex points are the same [5]. We also note that the system has the invariant quantity $\sum_{m=1}^{N} \cos\Theta_m$ due to the invariance of the Hamiltonian with respect to the rotation around the pole.

The N-vortex system attracts many researchers as a nonlinear Hamiltonian dynamical system [12]. For instance, the system of the three vortex points is integrable in the absence of the pole vortices and its motion was studied well [5,6,15]. Many relative fixed configurations of the N vortex points were systematically found [10,11,13]. Relative periodic orbits were also determined by the invariance of the Hamiltonian under the action of groups [9,19]. In the meantime, when the N vortex points are spaced equally along the line of latitude, the polygonal ring configuration is called the "N-ring". The motion of the N-ring has been investigated in particular, since the ring configuration of the vortex structure is often observed in the numerical research of the atmospheric phenomena [4,14,16]; The linear and nonlinear stability analysis of the N-ring with and without the pole vortices was given [1–3,8,17]. The unstable motion of the perturbed N-ring was investigated [17,18].

On the other hand, the N-vortex problem appears when the Euler equations are solved by the vortex method; Discretizing the vorticity region at the initial time with a cluster of vortex points, we investigate the evolution of the vortex points as an approximated solution of the Euler equations. In order to attain an accurate approximation, the number of discretizing vortex points must be very large. However, mathematical analysis of the vortex-points system gets more difficult in general as the number of the vortex points increases. Therefore we sometimes reduce the N-vortex system to low-dimensional systems by assuming a certain symmetry, and then study them as embedded subsystems. For instance, in the papers [17] and [18], the N-vortex system was successfully reduced to the integrable two-dimensional systems, with which the existence of the periodic, the heteroclinic and the homoclinic orbits and their stability were investigated. Thus the reduced systems help us understand the dynamics of the large number of N vortex points.

In the article, we give a reduction method of the N-vortex system to invariant dynamical systems. In Section 2, the linear stability analysis of the N-ring [17,18] is reviewed. It is required to characterize the invariant systems in the following sections. In Section 3, we show that it is possible to reduce the system by considering the invariant property for a shift transformation. The reduced dynamical system exists for every factor p of N, and it is equivalent to the p-vortex system generated by the averaged Hamiltonian on the sphere when the strengths of the pole vortices are modified suitably. In Section 4, we reduce the N-vortex system by the invariance with respect to a pole reversal transformation when the strength of the north pole vortex is equivalent to that of the south pole vortex. We conclude and discuss the results in the last section.

2. Preliminary results

We give a brief summary of the linear stability analysis for the N-ring [17,18], which is expressed by

$$\Theta_m = \theta_0, \quad \Psi_m = \frac{2\pi m}{N}, \quad m = 1, 2, \ldots, N. \tag{4}$$

The N-ring is a relative equilibrium for the Eqs. (1) and (2) rotating with the constant velocity $V_0(N)$ in the longitudinal direction,

$$V_0(N) = \frac{\Gamma_1 - \Gamma_2}{4\pi \sin^2\theta_0} + \frac{(\Gamma_1 + \Gamma_2 + 2\pi)\cos\theta_0}{4\pi \sin^2\theta_0} - \frac{1}{2N}\frac{\cos\theta_0}{\sin^2\theta_0}.$$

When we add small perturbations to the equilibrium,

$$\begin{aligned} \Theta_m(t) &= \theta_0 + \epsilon\theta_m(t), \\ \Psi_m(t) &= \frac{2\pi m}{N} + V_0(N)t + \epsilon\varphi_m(t), \quad |\epsilon| \ll 1, \end{aligned} \tag{5}$$

we obtain the linearized equations of $O(\epsilon)$ for the perturbations:

$$\dot{\theta}_m = \frac{1}{2N\sin\theta_0} \sum_{j\neq m}^{N} \frac{\varphi_m - \varphi_j}{1 - \cos\frac{2\pi}{N}(m-j)}, \tag{6}$$

$$\dot{\varphi}_m = \frac{1}{2N\sin^3\theta_0} \sum_{j\neq m}^{N} \frac{\theta_m - \theta_j}{1 - \cos\frac{2\pi}{N}(m-j)} + B_N\theta_m. \tag{7}$$

The parameter B_N is denoted by

$$B_N = \frac{1+\cos^2\theta_0}{2N\sin^3\theta_0} - \frac{\kappa_1(1+\cos^2\theta_0)}{2\sin^3\theta_0} - \frac{\kappa_2\cos\theta_0}{2\sin^3\theta_0}, \tag{8}$$

in which κ_1 and κ_2 are defined by

$$\kappa_1 = \frac{\Gamma_1 + \Gamma_2 + 2\pi}{2\pi}, \quad \kappa_2 = \frac{\Gamma_1 - \Gamma_2}{\pi}.$$

Then, we have obtained the eigenvalues and their corresponding eigenvectors for the linearized equations (6) and (7).