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Linear and nonlinear front instabilities in bistable systems

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Abstract

The stability of planar fronts to transverse perturbations in bistable systems is studied using the Swift–Hohenberg model and an urban population model. Contiguous to the linear transverse instability that has been studied in earlier works, a parameter range is found where planar fronts are linearly stable but nonlinearly unstable; transverse perturbations beyond some critical size grow rather than decay. The nonlinear front instability is a result of the coexistence of stable planar fronts and stable large-amplitude patterns. While the linear transverse instability leads to labyrinthine patterns through fingering and tip splitting, the nonlinear instability often evolves to spatial mixtures of stripe patterns and irregular regions of the uniform states.

Keywords: Front instabilities; Pattern formation; Nonlinear dynamics; Swift–Hohenberg equation

1. Introduction

Pattern formation phenomena in bistable systems are determined to a large extent by front instabilities. Fronts which are bi-asymptotic to a pair of stable uniform states can go through transverse instabilities leading to stationary labyrinthine patterns, or through non-equilibrium Ising–Bloch (NIB) bifurcations resulting in traveling wave phenomena such as Bloch spiral waves. The coupling of the two type of instabilities can induce irregular spatio-temporal behaviors ("Bloch-front turbulence") involving recurrent events of vortex-pair nucleation and annihilation. Labyrinthine patterns arising from transverse front instabilities have been observed in the FIS reaction [1] and in the periodically forced oscillatory Belousov–Zhabotinsky (BZ) reaction [2]. The forcing in this case was provided by periodic uniform illumination at a frequency twice as large as the system's oscillation frequency (2:1 forcing). Bloch spiral waves have been observed in the periodically forced BZ reaction and in liquid crystals [3,4]. Recent experiments on the periodically forced BZ reaction have also demonstrated Bloch-front turbulence [5]. These front instabilities have been found and analyzed in various models including the FitzHugh–Nagumo (FHN) model and a variant of the complex Ginzburg–Landau equation (FCGL) that describes 2:1 periodic forcing of uniform oscillations [6–9].

Another factor affecting pattern formation in bistable systems is the possible pinning of fronts between a pattern and a homogeneous state. Studies of the Swift–Hohenberg (SH) model in one space dimension showed that self-induced pinning, due to the oscillatory shape of the front tails, may prevent a front between a patterned state and a uniform state from propagating [10,11]. The result is that the evolution of a pattern in the SH model might not result in a final state with the lowest free energy.

Bistable systems often arise as a result of symmetry breaking instabilities of uniform states. This is the case with the FHN

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and the SH models where uniform states lose stability in pitchfork bifurcations. A different case is the FCGL equation for 2:1 forcing. The unforced oscillations appear in a Hopf bifurcation of a stationary uniform solution and correspond to a continuous family of solutions whose phases span the whole circle. The 2:1 forcing induces a pair of saddle-node bifurcations which fix the oscillation phases at two stable values shifted by π with respect to one another. Quite often the stationary uniform states, undergoing the zero-wavenumber pitchfork or Hopf bifurcations, go through finite-wavenumber instabilities as well. Resonant coupling of the zero and finite-wavenumber modes can lead to large amplitude patterns [12–15] observed in various systems [16–19].

In this paper we introduce and study another possible outcome of the coupling between zero and finite-wavenumber instabilities — a nonlinear transverse front instability. The asymptotic patterns that develop differ from those developing from linear transverse instabilities in that they often contain regions of uniform states coexisting with stripe domains. We first demonstrate the nonlinear front instability in two different models, the SH equation and a population model [20] (Section 2). We then use the SH equation to study both the linear and nonlinear transverse front instabilities and map them along the bifurcation parameter axis (Section 3). We also find (Section 4) that depinning of a front between a homogeneous state and a pattern occurs via a zigzag instability mechanism, which works far more efficiently than one-dimensional nucleation [10] and greatly reduces the pinning range.

2. Numerical demonstrations of linear and nonlinear transverse front instabilities

We consider here two examples of bistable systems, the SH equation and a population model describing urban segregation phenomena [20]. In both models front solutions bi-asymptotic to a symmetric pair of uniform states can become linearly unstable to transverse perturbations. The asymptotic patterns resulting from these linear transverse instabilities are stationary labyrinthine patterns as found in other models such as the FHN and FCGL. Contiguous to these instabilities in parameter space, however, there exist parameter ranges where the fronts are linearly stable but finite-size transverse perturbations still grow. Depending on initial conditions, the asymptotic patterns in this case may look like labyrinths that develop from linear instabilities, or mixtures of stripes and regions of the two uniform states. We demonstrate these behaviors by numerically solving the SH equation and the population model.

2.1. The Swift–Hohenberg equation

The SH equation we consider has the form [10,21]

$$u_t = \epsilon u - (\nabla^2 + 1)^2 u - u^3, \qquad (1)$$

where u is a real scalar field and ϵ is the bifurcation control parameter. The zero solution $u = 0$ loses stability to finite-wavenumber perturbations at $\epsilon = 0$, and goes through a

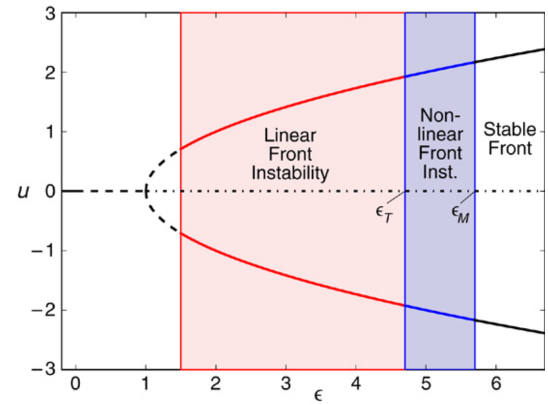

Fig. 1. Bifurcation diagram of uniform solutions to the Swift–Hohenberg Eq. (1). The solution $u = 0$ for $\epsilon < 0$ becomes unstable to finite wavenumber perturbations at $\epsilon = 0$ and then bifurcates to a pair of nonzero unstable solutions at $\epsilon = 1$. At $\epsilon = 1.5$ the two nonzero solutions stabilize but fronts between the two solutions have a linear transverse instability. For $\epsilon \in [\epsilon_T, \epsilon_M]$ fronts are linearly stable to transverse perturbations but large perturbations grow and create a patterned state. Above ϵ_M the fronts are globally stable.

pitchfork bifurcation at $\epsilon = 1$. The two uniform states, $u_\pm = \pm\sqrt{\epsilon - 1}$, that appear above $\epsilon = 1$ are unstable to finite-wavenumber perturbations but become stable above $\epsilon = 3/2$. Fig. 1 shows a bifurcation diagram of the uniform solutions and the finite-wavenumber instabilities they go through (with additional thresholds to be discussed below).

The bistability of uniform states in the range $\epsilon > 3/2$ allows for front solutions approaching $u_\pm$ asymptotically as $x \to \pm\infty$ or $x \to \mp\infty$. These front solutions are linearly unstable to transverse perturbations up to a threshold $\epsilon = \epsilon_T$ to be calculated in the next section. This linear instability is demonstrated in Fig. 2(a). Beyond ϵ_T, the linear transverse instability disappears; small transverse perturbations of the front decay out as Fig. 2(b) shows. The front, however, remains unstable to finite-size perturbations, implying a *nonlinear* transverse instability. The instability is demonstrated in Fig. 2(c) which also shows the asymptotic pattern that develops — a spatial mixture of parallel stripes and regions of the two stable uniform states. The nonlinear transverse instability disappears at a yet higher threshold, ϵ_M, to be calculated in the next section. Fig. 2(d) demonstrates the global front stability above ϵ_M by showing the retraction of a pattern state to a planar front.

2.2. A population model

The population model we consider here has been introduced and studied in the context of segregation phenomena in residential neighborhoods [20]. It consists of three dynamical variables, u, v, and s, representing, respectively, the densities of two distinct populations and the socio-economic status. A simple version of the model equations, not including non-local migration, is [20]:

$$\begin{aligned} u_t &= u - u^2 + us + \nabla^2 u - \delta_1 \nabla^2 s, \\ v_t &= \alpha v - v^2 - \beta v s + \delta_2 \nabla^2 v + \delta_3 \nabla^2 s, \\ s_t &= \epsilon(u - \gamma v - \mu s) - \xi s^3 + \delta_4 \nabla^2 s. \end{aligned} \qquad (2)$$