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Numerical inverse scattering for the Korteweg–de Vries and modified Korteweg–de Vries equations

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ABSTRACT

Recent advances in the numerical solution of Riemann–Hilbert problems allow for the implementation of a Cauchy initial-value problem solver for the Korteweg–de Vries equation (KdV) and the defocusing modified Korteweg–de Vries equation (mKdV), without any boundary approximation. Borrowing ideas from the method of nonlinear steepest descent, this method is demonstrated to be asymptotically accurate. The method is straightforward for the case of defocusing mKdV due to the lack of poles in the Riemann–Hilbert problem and the boundedness properties of the reflection coefficient. Solving KdV requires the introduction of poles in the Riemann–Hilbert problem and more complicated deformations. The introduction of a new deformation for KdV allows for the stable asymptotic computation of the solution in the entire spacial and temporal plane. KdV and mKdV are dispersive equations, and this method can fully capture the dispersion with spectral accuracy. Thus, this method can be used as a benchmarking tool for determining the effectiveness of future numerical methods designed to capture dispersion. This method can easily be adapted to other integrable equations with Riemann–Hilbert formulations, such as the nonlinear Schrödinger equation.

1. Introduction

We consider the initial-value problem on the whole line for the Korteweg–de Vries equation (KdV)

$$\begin{aligned} &q_t + 6qq_x + q_{xxx} = 0, \\ &q(x,0) = q_0(x) \in \mathcal{S}(\mathbb{R}), \end{aligned} \tag{1.1}$$

where subscripts denote partial differentiation and $\mathcal{S}(\mathbb{R})$ is the Schwartz class on $\mathbb{R}$ [1, Definition 4.2.1]. We also consider the defocusing modified Korteweg–de Vries equation (mKdV), given by

$$\begin{aligned} &q_t - 6q^2 q_x + q_{xxx} = 0, \\ &q(x,0) = q_0(x) \in \mathcal{S}(\mathbb{R}). \end{aligned} \tag{1.2}$$

KdV describes the propagation of long waves in dispersive media, e.g., long surface water waves [2]. Historically, KdV is the first known case of a partial differential equation (PDE) that is solvable by the inverse scattering transform [3]. KdV and mKdV can also be thought of as dispersive regularizations of the Burgers and modified Burgers equations, respectively.

The presence of dispersion makes the approximation of solutions of KdV and mKdV through numerical methods especially difficult; see Section 2 for a detailed discussion. To see this qualitatively, in Fig. 1 we approximate the solution of KdV with $q(x,0) = A\,\mathrm{sech}^2(x)$, where $A = 3.2$ using the numerical scheme presented here. With $A = 3$, the solution would be a two-soliton solution without any dispersive tail [4]. Notice that a significant dispersive tail forms even though the solution is close to the soliton case. The issue becomes worse when we consider solutions that are farther from a soliton solution; see Fig. 2.

To combat this dispersive complication, we exploit the integrability of KdV and mKdV and evaluate the inverse scattering transform (IST) numerically. It is important to note that more conventional methods are also applicable to other dispersive equations that may not be integrable, whereas our method requires integrability. Computing the IST involves developing techniques to compute the forward transform (direct scattering) and the inverse transform (inverse scattering). Our approach to direct scattering employs collocation methods and existing spectrum approximation techniques. For inverse scattering we use the numerical method for Riemann–Hilbert problems (RHPs) presented in [5]. After deforming the RHP in the spirit of Deift

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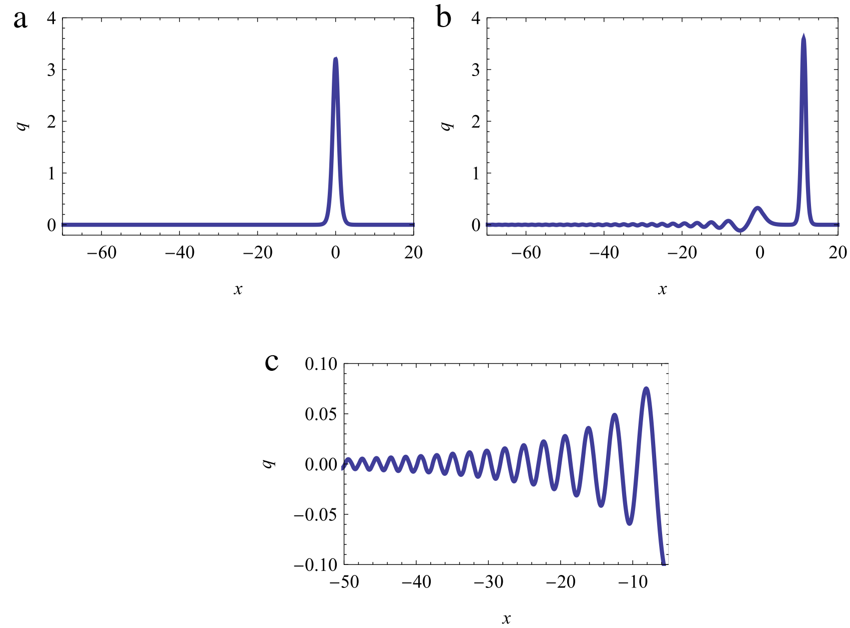

Fig. 1. Numerical solution of KdV with initial data that is close to a two-soliton solution. (a) Initial condition. (b) Solution at $t = 1.5$. (c) Dispersive tail at $t = 1.5$.

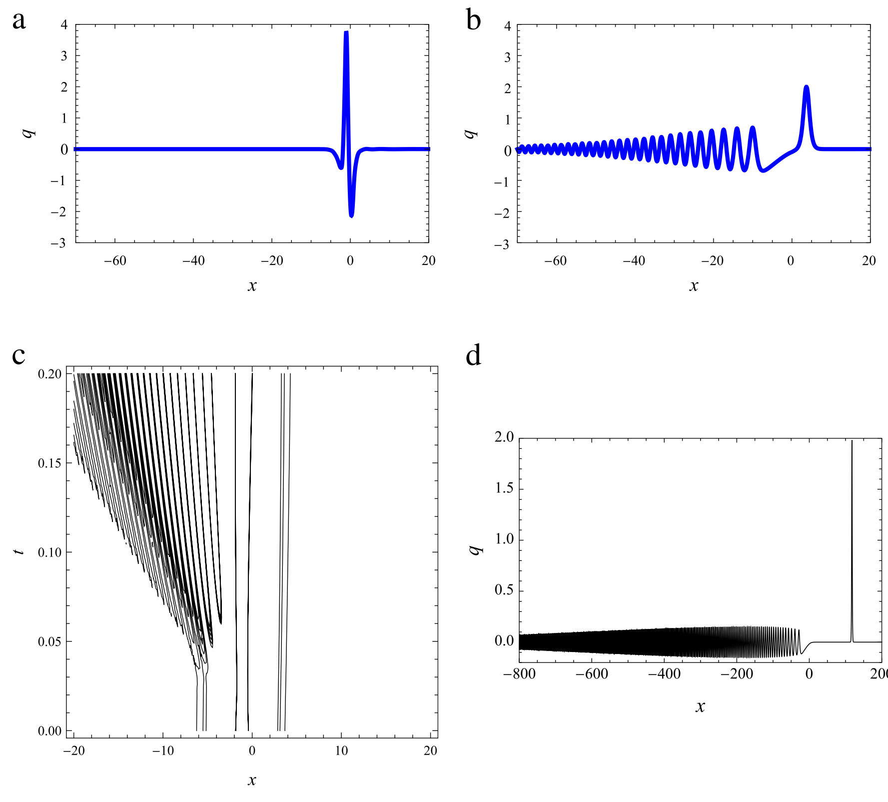

Fig. 2. Numerical solution of KdV which is far from a pure soliton solution. (a) Initial condition obtained by adding a soliton to the Riemann–Hilbert problem associated with $q(x, 0) = -2.3\,\text{sech}^2(x)$. (b) Solution at $t = 1.5$. (c) A contour plot showing the birth of the dispersive tail. (d) Solution at $t = 30$. It is not practical to use conventional methods to capture this solution for longer times.

and Zhou [6–8], the numerical method becomes asymptotically accurate: the work required to compute the solution at a point to a desired accuracy is bounded for all x and t. In this method the roles of x and t are reduced to that of parameters. No time-stepping or spatial discretization is needed, and the code could easily be run in parallel.

We start at points bounded away from with background material concerning RHPs and their numerical solution. The numerical direct and inverse scattering for defocusing mKdV is then presented along with numerical results. The RHP for mKdV has a simple form and the deformations are straightforward. Next, KdV is considered. Now one has to deal with the addition of solitons to the problem. After deformation, the RHP for KdV has a singularity, and this requires two additional deformations. We introduce a new deformation that is not present, to our knowledge, in the existing literature. This new transition region allows for