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Developing homogeneous isotropic turbulence

Wouter J.T. Bos [a], Colm Connaughton [b,*], Fabien Godeferd [a]

[a] Laboratoire de Mécanique des Fluides et d'Acoustique, CNRS UMR 5509, École Centrale de Lyon, Université de Lyon, France

[b] Mathematics Institute and Centre for Complexity Science, University of Warwick, Coventry CV4 7AL, UK

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ABSTRACT

We investigate the self-similar evolution of the transient energy spectrum, which precedes the establishment of the Kolmogorov spectrum in homogeneous isotropic turbulence in three dimensions using the EDQNM closure model. The transient evolution exhibits self-similarity of the second kind and has a non-trivial dynamical scaling exponent, which results in the transient spectrum having a scaling that is steeper than the Kolmogorov $k^{-5/3}$ spectrum. Attempts to detect a similar phenomenon in DNS data are inconclusive, owing to the limited range of scales available.

1. Introduction to transient spectra in turbulence

Although a large amount of work has been done characterising the properties of the Kolmogorov $k^{-5/3}$ spectrum of three dimensional turbulence, rather less attention has been paid to the transient evolution that leads to its establishment. This transient evolution is essentially non-dissipative, since it describes the cascade process before it reaches the dissipation scale. Part of the reason why this process has attracted relatively little attention is that this transient evolution is very fast, typically taking place within a single large eddy turnover time. It is thus of little relevance to the developed turbulence regime of interest in many applications. Nevertheless, one may ask whether this developing turbulence, as one might call this transient regime, displays any interesting scaling properties. Previous studies of the developing regime in weak magnetohydrodynamic (MHD) turbulence [1] suggest that this transient regime might have non-trivial scaling properties; in this case it was found that the establishment of the Kolmogorov spectrum is preceded by a transient spectrum that is steeper than the Kolmogorov spectrum. The latter is, in turn, set up from right to left in wavenumber space only after the transient spectrum has reached the end of the inertial range and has started to produce dissipation.

Subsequent studies suggest that this behaviour, in particular the occurrence of a non-trivial dynamical scaling exponent, is typical for turbulent cascades which are of finite capacity—meaning that the stationary spectrum can only contain a finite amount of energy. The Kolmogorov spectrum of three dimensional turbulence is in the class of finite capacity systems, as we shall see below. There are, however, examples of other turbulent cascades which are not; infinite capacity cascades are common in wave turbulence for example [2]. In addition to the MHD cascade mentioned above, examples of non-trivial scaling exponents in finite capacity cascades have been found in developing wave turbulence [3,4], Bose–Einstein condensation [5,6] and cluster–cluster aggregation [7]. Although a possible heuristic explanation of the transient scaling in the MHD context has been put forward in [8], this heuristic relies heavily on the anisotropy of the MHD cascade and does not seem readily generalisable to other contexts. In general, the transient exponent is associated with a self-similarity problem of the second kind [9]. From a mathematical point of view, its solution requires solving a nonlinear equation in which the exponent appears as a parameter which is fixed by requiring consistency with boundary conditions. It is probably unrealistic to expect that there is a general heuristic argument capable of resolving such a mathematically challenging problem. This is not to say, however, that particular cases may not be amenable to heuristic arguments that take into account the underlying *physical* mechanisms driving the transient evolution rather than taking a purely mathematical point of view.

This issue has not yet been studied in the context of homogeneous isotropic turbulence. Investigations of transient spectra in the classical Leith closure model [10] have suggested, however, that the transient spectrum of developing homogeneous isotropic turbulence is indeed non-trivially steeper than $k^{-5/3}$ [11]. In this work, we investigate the transient evolution of homogeneous isotropic turbulence using the eddy-damped quasi-normal Markovian (EDQNM) closure model and direct numerical simulation (DNS) of the Navier–Stokes equation.

The transient spectrum might be expected to evolve self-similarly. In other words there is a typical wavenumber, $s(t)$, which

* Corresponding author. Tel.: +44 2476 523572; fax: +44 24 7652 4182.
E-mail addresses: Wouter.Bos@ec-lyon.fr (W.J.T. Bos), connaughtonc@gmail.com, C.P.Connaughton@warwick.ac.uk (C. Connaughton), Fabien.Godeferd@ec-lyon.fr (F. Godeferd).

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grows in time, and a dynamical scaling exponent, a, such that

$$E_k(t) \asymp c s(t)^a F(\xi) \quad \text{where } \xi = \frac{k}{s(t)}. \tag{1}$$

Here $\asymp$ denotes the scaling limit: $k \to \infty, s(t) \to \infty$ with ξ fixed and c is an order unity constant which ensures that $E_k(t)$ has the correct physical dimensions, L^3T^{-2}. As we shall see, if the exponent, $(5+a)/2$, is greater than 1, then the characteristic wavenumber diverges in finite time corresponding to a cascade which accelerates "explosively". The direct cascade in 3D turbulence is of this type. The characteristic wavenumber is most easily defined as a ratio of moments of the energy spectrum. Let us define

$$M_n(t) = \int_0^\infty k^n E_k(t)\, dk. \tag{2}$$

Eq. (1) suggests that the ratio $M_{n+1}(t)/M_n(t)$ is proportional to $s(t)$ so that we may define a typical scale intrinsically by

$$s_n(t) = \frac{M_{n+1}(t)}{M_n(t)}. \tag{3}$$

A little caution is required: we must take n sufficiently high to ensure that the moments $M_n(t)$ used in defining the typical scale, converge at zero. Otherwise, the integral is dominated by the initial condition or forcing scale and does not capture the scaling behaviour. In this paper, we mostly take $n = 2$, which turns out to be sufficient for our purposes, although we will compare the behaviour obtained for $n = 2$ and $n = 3$ in our numerical simulations to assure the reader that the picture is consistent.

We would like to emphasise that the self-similar transient dynamics which we study in this paper occur *before* the onset of dissipation. This is in contrast to the transient dynamics describing the long time decay of homogeneous isotropic turbulence *after* the onset of dissipation which are also believed to exhibit self-similarity. See [12] for recent experiments and a review of previous work. Some numerical results on the long time transient dynamics of the EDQNM model can be found in [13]. The pre-dissipation transient occurs very quickly. Indeed, as we shall see, the typical scale, $s(t)$, in this regime diverges as $s(t) \sim (t^*-t)^b$ where t^* is the time at which the onset of dissipation occurs (typically less than a single turnover time) and $b < 0$. For finite Reynolds number, this singularity is regularised by the finiteness of the dissipation scale. The fact that, in the limit of infinite Reynolds number, the typical scale can grow by an arbitrary amount in an arbitrarily small time interval as t^* is approached explains the statement often found in the literature that the Kolmogorov spectrum is established quasi-instantaneously in the limit of large Reynolds number.

2. The EDQNM model

In this section we examine the self-similar solutions of the EDQNM model [14]. The structure of the EDQNM model can be obtained in different ways. One way is starting from the Quasi-Normal assumption [15]. Another way is by simplifying the Direct Interaction Approximation [16] which was obtained by applying a renormalised perturbation procedure to the Navier–Stokes equation. It is thus directly related to the Navier–Stokes equation, unlike the Leith model which was heuristically proposed to capture some features of the nonlinear transfer in isotropic turbulence. However, recent work [17] showed that the structure of the Leith model can be obtained by retaining a subset of triad interactions involving elongated triads from closures like EDQNM. EDQNM containing a wider variety of triad interactions, it is able to capture more details of the actual dynamics of Navier–Stokes turbulence, as for example illustrated in [18]. At the same time it has the advantage over DNS that much higher Reynolds numbers can be obtained.

The EDQNM model closes the Lin equation by expressing the nonlinear triple correlations as a function of the energy spectrum,

$$\frac{\partial E_k}{\partial t} = \mathrm{T}[E_k] - 2\nu k^2 E_k \tag{4}$$

where ν is the viscosity and $\mathrm{T}[E_k]$ represents the nonlinear interactions between different scales. $\mathrm{T}[E_k]$ has the form

$$\mathrm{T}[E_k] = \int_\Delta dk_1 dk_2 T_{k,k_1,k_2} k (k_1 k_2)^{-1} E_{k_2} (k^2 E_{k_1} - k_1^2 E_k), \tag{5}$$

where Δ signifies that the region of integration is over all values of k_1 and k_2 for which the triad (k, k_1, k_2) can form the sides of a triangle and the interaction strength of each triad, T_{k,k_1,k_2}, is given by

$$T_{k,k_1,k_2} = \frac{k_1}{k}(\theta_k \theta_{k_1} + \theta_{k_2}^3) \frac{1 - \exp\left[-(\mu_k + \mu_{k_1} + \mu_{k_2})t\right]}{\mu_k + \mu_{k_1} + \mu_{k_2}} \tag{6}$$

where θ, θ_1 and θ_2 are the cosines of the angles opposite to k, k_1 and k_2 respectively in the triangle formed by the triad (k, k_1, k_2) and

$$\mu_k = \nu k^2 + \lambda \sqrt{\int_0^k p^2 E_p\, dp}, \tag{7}$$

is the timescale associated with an eddy at wavenumber k, parameterised by the EDQNM parameter, λ, which is chosen equal to 0.49, [19]. For a full discussion of the origins and properties of the EDQNM model see [20,21]. We concern ourselves here only with the inviscid limit where $\nu \to 0$.

If we substitute the scaling ansatz, Eq. (1) into Eq. (4) with $\nu = 0$ then, in the scaling limit, the nonlinear transfer term becomes homogeneous of degree $\frac{3+3a}{2}$ in s and one finds

$$\frac{ds}{dt} = \sqrt{c}\, s^{\frac{5+a}{2}} \tag{8}$$

$$aF - \xi \frac{dF}{d\xi} = \mathrm{T}[F]. \tag{9}$$

Scaling alone does not determine the dynamical exponent a. To determine a we may attempt to impose conservation of energy on the scaling solution to obtain a second constraint which will fix a. Let us go down this path, at first naively, and then reconsider our argument more carefully:

1. *Forced case.*
   If we consider forced turbulence, then energy is injected into the system in a narrow band of low wavenumbers (which necessarily lie outside of the region of applicability of the scaling solution). The total energy grows linearly in time (remember we are interested in the dynamics *before* the onset of dissipation): $\int_0^\infty E_k(t)\, dk = \epsilon t$. If we use the scaling ansatz, Eq. (1), differentiate with respect to time and rearrange we obtain

$$\frac{ds}{dt} = \epsilon \left[(a+1)c \int_0^\infty F(\xi)\, d\xi\right]^{-1} s^{-a}. \tag{10}$$

   Taken together with Eq. (8) we are led to expect

$$a = -\frac{5}{3} \quad \text{for forced turbulence.} \tag{11}$$

   The same conclusion would be reached by dimensional analysis of Eq. (1) under the assumption that the sole parameter available is the energy flux, ϵ, (having physical dimension L^2T^{-3}).