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Effective computation of the multivariable Alexander polinomial of Lorenz links

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ABSTRACT

Given two different representations of a Lorenz link, we compare how they affect the computation of the multivariable Alexander polynomial. We also compare the Alexander polynomial with the trip number and genus. Our experimental results lead us to conjecture that, for Lorenz knots, the Alexander polynomial is an equivalent invariant to the pair (trip number, genus). Finally, we give a counterexample in the case of Lorenz links.

1. Introduction

We define a *Lorenz flow* as a semi-flow that has a singularity of saddle type with a one-dimensional unstable manifold and an infinite set of hyperbolic periodic orbits, whose closure contains the saddle point (see [7]). A Lorenz flow, together with an extra geometric assumption (see [11]) is called a *Geometric Lorenz flow*. The dynamics of this type of flows can be described by first-return one-dimensional maps with one discontinuity, that are not necessarily surjective in the continuity subintervals. This maps are called Lorenz maps, more precisely, we will adopt the following definition introduced in [7].

Definition 1. Let $P < 0 < Q$ and $r \geq 1$. A C^r Lorenz map $f : [P, Q] \setminus \{0\} \to [P, Q]$ is a map described by a pair (f_-, f_+),

$$f(x) = \begin{cases} f_-(x) & \text{if } P \leq x < 0 \\ f_+(x) & \text{if } 0 < x \leq Q, \end{cases}$$

where:

(1) $f_- : [P, 0] \to [P, Q]$ and $f_+ : [0, Q] \to [P, Q]$are continuous and strictly increasing maps;
(2) $f(P) = P, f(Q) = Q$ and f has no other fixed points in $[P, Q] \setminus \{0\}$.
(3) There exists $\rho > 0$, the exponent of f, such that

$$f_-(x) = \widetilde{f}_-(|x|^\rho) \quad \text{and} \quad f_+(x) = \widetilde{f}_+(|x|^\rho)$$

where $\widetilde{f}_-$ and $\widetilde{f}_+$, the coefficients of the Lorenz map, are C^r diffeomorphisms defined on appropriate closed intervals.

This Lorenz map is denoted by (P, Q, f_-, f_+) (if there is no ambiguity about the interval of definition, we erase the corresponding symbols P, Q).

Note that both f_- and f_+ are defined in 0, because of this ambiguity we consider the map undefined in 0.

Let $f^j = f \circ f^{j-1}, f^0 = id$, be the j-th iterate of the map f. We define the *itinerary* of a point x under a Lorenz map f as $i_f(x) = (i_f(x))_j, j = 0, 1, \ldots$, where

$$(i_f(x))_j = \begin{cases} L & \text{if} \quad f^j(x) < 0 \\ 0 & \text{if} \quad f^j(x) = 0 \\ R & \text{if} \quad f^j(x) > 0. \end{cases}$$

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It is obvious that the itinerary of a point x will be a finite sequence in the symbols L and R with 0 as its last symbol, if and only if x is a pre-image of 0 and otherwise it is one infinite sequence in the symbols L and R. So it is natural to consider the symbolic space Σ of sequences $X_0 \cdots X_n$ on the symbols $\{L, 0, R\}$, such that $X_i \neq 0$ for all $i < n$ and: $n = \infty$ or $X_n = 0$, with the lexicographic order relation induced by $L < 0 < R$.

It is straightforward to verify that, for all $x, y \in [P, Q]$, we have

(1) If $x < y$ then $i_f(x) \le i_f(y)$, and
(2) If $i_f(x) < i_f(y)$ then $x < y$.

We define the *kneading invariant* associated to a Lorenz map $f = (f_-, f_+)$, as

$$K_f = (K_f^-, K_f^+) = (Li_f(f_-(0)), Ri_f(f_+(0))).$$

We say that a pair $(X, Y) \in \Sigma \times \Sigma$ is *admissible* if $(X, Y) = K_f$ for some Lorenz map f.

Consider the *shift map* $s : \Sigma \setminus \{0\} \to \Sigma$, $s(X_0 \cdots X_n) = X_1 \cdots X_n$. The set of admissible pairs is characterized, combinatorially, in the following way (see for example [6]).

Proposition 1. *A pair $(X, Y) \in \Sigma \times \Sigma$ is admissible if and only if $X_0 = L$, $Y_0 = R$ and, for $Z \in \{X, Y\}$ we have:*
(1) If $Z_i = L$ then $s^i(Z) \le X$;
(2) If $Z_i = R$ then $s^i(Z) \ge Y$; with strict inequality (1) (resp. (2)) if X (resp. Y) is finite.

2. Braids, Lorenz links and the Alexander polynomial

Let $n > 0$ be an integer. We denote by B_n the braid group on n strings given by the following presentation (see [1]):

$$B_n = \left\langle \sigma_1, \sigma_2, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{ll} \sigma_i\sigma_j = \sigma_j\sigma_i & (|i-j| \ge 2) \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} & (i = 1, \ldots, n-2) \end{array} \right\rangle.$$

Where σ_i denotes a crossing between the strings occupying positions i and $i+1$, such that the string in position i crosses (in the up to down direction) over the other, analogously σ_i^{-1}, the algebraic inverse of σ_i, denotes the crossing between the same strings, but in the negative sense, i.e., the string in position i crosses under the other. A *positive braid* is a braid with only positive crossings. A simple braid is a positive braid such that each two strings cross each other at most once. So there is a canonical bijection between the permutation group Σ_n and the set S_n, of simple braids with n strings, which associates to each permutation π, the braid b_π, where each point i is connected by a straight line to $\pi(i)$, keeping all the crossings positive.

Let X be a periodic sequence with least period k, and let $\varphi \in \Sigma_k$ be the permutation that associates to each i the position occupied by $s^i(X)$ in the lexicographic ordering of the k-tuple $(s(X), \ldots, s^k(X))$ $(s^k(X) = X)$. Define $\pi \in \Sigma_k$ to be the permutation given by $\pi(\varphi(i)) = \varphi(i \bmod k + 1)$, i.e., $\pi(i) = \varphi(\varphi^{-1}(i) \bmod k + 1)$. We associate to π the corresponding simple braid $b_\pi \in B_k$ and call it the *Lorenz braid* associated to X. Since X is periodic, this braid represents a knot, and we call it the *Lorenz knot* associated to X. The same method is valid if we consider a pair of sequences. We obtain in this case a Lorenz braid which represents a *Lorenz link*. The Lorenz braid produced in this way is just one possible representative for the respective Lorenz link.

Example. Let $Z = (LRRLRRLRRLR)^\infty$. Hence we have $s^{11}(Z) = Z$, $s(Z) = (RRLRRLRRLRL)^\infty$, $s^2(Z) = (RLRRLRRLRLR)^\infty$, $s^3(Z) = (LRRLRRLRLRR)^\infty$, $s^4(Z) = (RRLRRLRLRRL)^\infty, \ldots$. Now after lexicographic reordering the $s^i(Z)$ we obtain $s^9(Z) < s^6(Z) < s^3(Z) < s^{11}(Z) < s^8(Z) < s^5(Z) < s^2(Z) < s^{10}(Z) < s^7(Z) < s^4(Z) < s^1(Z)$ and $\varphi = (1, 11, 4, 10, 8, 5, 6, 2, 7, 9)$ written as a disjoint cycle. Finally we obtain $\pi = (1, 8, 4, 11, 7, 3, 10, 6, 2, 9, 5)$ and the braid represented in Fig. 1.

$$b_\pi = \sigma_4\sigma_5\sigma_6\sigma_7\sigma_8\sigma_9\sigma_{10}\sigma_3\sigma_4\sigma_5\sigma_6\sigma_7\sigma_8\sigma_9\sigma_2\sigma_3\sigma_4\sigma_5\sigma_6\sigma_7\sigma_8\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5\sigma_6\sigma_7.$$

Given an admissible pair (X, Y) of symbolic sequences, there is another way of producing a braid representing the Lorenz link associated to (X, Y). This method was developed by Birman and Williams (see [2]) and contains an explicit formula to compute a reduced braid which represents (as a closed braid) the same Lorenz link.

The *trip number*, t, of a finite sequence X, is the number of syllables in X, a syllable being a maximal subword of X, of the form L^aR^b. The trip number of an admissible pair of sequences is the sum of the two trip numbers of the two sequences.

Birman and Williams conjectured in [2] that, for the case of a Lorenz knot τ, $b(\tau) = t(\tau)$, where $b(\tau)$ is the braid index of the finite sequence associated to τ. In [10], following a result obtained by Franks and Williams in [5], Waddington observed that this conjecture is true.

The Birman–Williams formula is obtained in the following way. Let $\pi \in S_n$ be a Lorenz permutation. So we construct the Birman–Williams braid $b_{BW}(\pi)$ (or simply b_{BW} if there is no risk of confusion) associated to π in the following way:

$$b_{BW}(\pi) = \Delta_n^2 \prod_{i=1}^{t-1} (\sigma_1 \cdots \sigma_i)^{n_i} \prod_{i=t-1}^{1} (\sigma_{t-1} \cdots \sigma_i)^{m_{t-i}}$$

where t is the trip number, the exponents are given by

$$n_i = card\{j \text{ such that } \pi(j) - j = i + 1 \text{ and } \pi(j) < \pi^2(j)\}$$
$$m_i = card\{j \text{ such that } j - \pi(j) = i + 1 \text{ and } \pi(j) > \pi^2(j)\}$$

and $\Delta_n \in B_n$ is the simple braid such that each two strings cross each other exactly once. It can be written, in terms of generators, in the following way:

$$\Delta_n = (\sigma_1 \cdots \sigma_{n-1})(\sigma_1 \cdots \sigma_{n-2}) \cdots \sigma_1\sigma_2\sigma_1.$$