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Review

Boolean delay equations: A simple way of looking at complex systems

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ABSTRACT

Boolean Delay Equations (BDEs) are semi-discrete dynamical models with Boolean-valued variables that evolve in continuous time. Systems of BDEs can be classified into *conservative* or *dissipative*, in a manner that parallels the classification of ordinary or partial differential equations. Solutions to certain conservative BDEs exhibit growth of complexity in time; such BDEs can be seen therefore as metaphors for biological evolution or human history. Dissipative BDEs are structurally stable and exhibit multiple equilibria and limit cycles, as well as more complex, fractal solution sets, such as Devil's staircases and "fractal sunbursts." All known solutions of dissipative BDEs have stationary variance. BDE systems of this type, both free and forced, have been used as highly idealized models of climate change on interannual, interdecadal and paleoclimatic time scales. BDEs are also being used as flexible, highly efficient models of colliding cascades of loading and failure in earthquake modeling and prediction, as well as in genetics. In this paper we review the theory of systems of BDEs and illustrate their applications to climatic and solid-earth problems. The former have used small systems of BDEs, while the latter have used large hierarchical networks of BDEs. We moreover introduce BDEs with an infinite number of variables distributed in space ("partial BDEs") and discuss connections with other types of discrete dynamical systems, including cellular automata and Boolean networks. This research-and-review paper concludes with a set of open questions.

1. Introduction

BDEs constitute a modeling framework especially tailored for the mathematical formulation of conceptual models of systems that exhibit threshold behavior, multiple feedbacks and distinct time delays [1–4]. BDEs are intended as a heuristic first step on the way to understanding problems too complex to model using systems of partial differential equations at the present time. One hopes, of course, to be able to eventually write down and solve the exact equations that govern the most intricate phenomena. Still, in the geosciences as well as in the life sciences and other natural sciences, much of the preliminary discourse is often conceptual.

BDEs offer a formal mathematical language that may help to bridge the gap between qualitative and quantitative reasoning. Besides, they are fun to play with and produce beautiful fractals by simple, purely deterministic rules. Furthermore, they also provide an unconventional view on the concepts of non-linearity and complexity.

In a hierarchical modeling framework, simple conceptual models are typically used to present hypotheses and capture isolated mechanisms, while more detailed models try to simulate the phenomena more realistically, and test for the presence and effect of the suggested mechanisms by direct confrontation with observations [5]. BDE modeling may be the simplest representation of the relevant physical concepts. At the same time, new results obtained with a BDE model often capture phenomena not yet found by using conventional tools [6–8]. BDEs suggest possible mechanisms that may be investigated using more complex models once their "blueprint" is detected in a simple conceptual model. As the study of complex systems garners increasing attention and is applied to diverse areas — from

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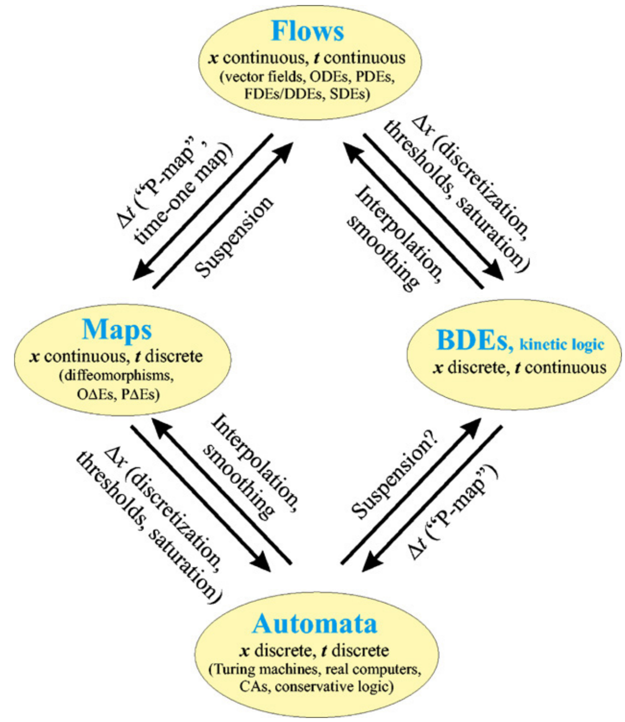

Fig. 1. The place of BDEs within dynamical system theory. Note the links: the discretization of t can be achieved by the Poincaré map (P-map) or a time-one map, leading from **Flows** to **Maps**. The opposite connection is achieved by suspension. To go from **Maps** to **Automata** we use the discretization of x. Interpolation and smoothing can lead in the opposite direction. Similar connections lead from **BDEs** to **Automata** and to **Flows**, respectively. Modified after Mullhaupt [2].

microbiology to the evolution of civilizations, passing through economics and physics — related Boolean and other discrete models are being explored more and more [9–13].

The purpose of this research-and-review paper is threefold: (i) summarize and illustrate key properties and applications of BDEs; (ii) introduce BDEs with an infinite number of variables; and (iii) explore more fully, connections between BDEs and other types of *discrete dynamical systems* (dDS). Therefore, we first describe the general form and main properties of BDEs and place them in the more general context of dDS, including cellular automata and Boolean networks (Section 2). Next, we summarize some applications, to climate dynamics (Section 3) and to earthquake physics (Section 4); these applications illustrate both the beauty and usefulness of BDEs. In Section 5 we introduce BDEs with an infinite number of variables, distributed on a spatial lattice ("partial BDEs") and point to several ways of potentially enriching our knowledge of BDEs and extending their areas of application. Further discussion and open questions conclude the paper (Section 6).

2. Boolean delay equations (BDEs)

BDEs may be classified as *semi-discrete dynamical systems*, where the variables are discrete — typically Boolean, *i.e.* taking the values 0 ("off") or 1 ("on") only — while time is allowed to be continuous. As such they occupy the previously "missing corner" in the rhomboid of Fig. 1, where dynamical systems are classified according to whether their time (t) and state variables (x) are continuous or discrete.

Systems in which both variables and time are continuous are called *flows* [14,15] (upper corner in the rhomboid of Fig. 1). Vector fields, ordinary and partial differential equations (ODEs and PDEs), functional and delay-differential equations (FDEs and DDEs) and stochastic differential equations (SDEs) belong to this category. Systems with continuous variables and discrete time (middle left corner) are known as *maps* [16,17] and include diffeomorphisms, as well as ordinary and partial difference equations (O△Es and P△Es).

In automata (lower corner) both the time and the variables are discrete; cellular automata (CAs) and all Turing machines (including real-world computers) are part of this group [10, 11,18], and so is the synchronous version of Boolean random networks [12,19]. BDEs and their predecessors, kinetic [20] and conservative logic, complete the rhomboid in the figure and occupy the remaining middle right corner.

The connections between flows and maps are fairly well understood, as they both fall in the broader category of *differentiable dynamical systems* (DDS [14–16]). Poincaré maps ("P-maps" in Fig. 1), which are obtained from flows by intersection with a plane (or, more generally, with a codimension-1 hyperplane) are standard tools in the study of DDS, since they are simpler to investigate, analytically or numerically, than the flows from which they were obtained. Their usefulness arises, to a great extent, from the fact that — under suitable regularity assumptions — the process of suspension allows one to obtain the original flow from its P-map; hence the properties of the flow can be deduced from those of the map, and vice-versa.

In Fig. 1, we have outlined by labeled arrows the processes that can lead from the dynamical systems in one corner of the rhomboid to the systems in each one of the adjacent corners. Neither the processes that connect the two dDS corners, automata and BDEs, nor these that connect either type of dDS with the adjacent-corner DDS — maps and flows, respectively — are as well understood as the (P-map, suspension) pair of antiparallel arrows that connects the two DDS corners. We return to the connection between BDEs and Boolean networks in Section 2.6 below. The key difference between kinetic logic and BDEs is summarized in the Appendix.

2.1. General form of a BDE system

Given a system with n continuous real-valued state variables $\mathbf{v} = (v_1, v_2, \dots, v_n) \in \mathbb{R}^n$ for which natural thresholds $q_i \in \mathbb{R}$ exist, one can associate with each variable $v_i \in \mathbb{R}$ a Boolean-valued variable, $x_i \in \mathbb{B} = \{0, 1\}$, *i.e.*, a variable that is either "on" or "off", by letting

$$x_i = \begin{cases} 0, & v_i \le q_i \\ 1, & v_i > q_i \end{cases}, \quad i = 1, \dots, n. \tag{1}$$

The equations that describe the evolution in time of the Boolean vector $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{B}^n$, due to the time-delayed interactions between the Boolean variables $x_i \in \mathbb{B}$ are of the form:

$$\begin{cases} x_1(t) = f_1\left[t, x_1(t-\theta_{11}), x_2(t-\theta_{12}), \dots, x_n(t-\theta_{1n})\right], \\ x_2(t) = f_2\left[t, x_1(t-\theta_{21}), x_2(t-\theta_{22}), \dots, x_n(t-\theta_{2n})\right], \\ \qquad \vdots \\ x_n(t) = f_n\left[t, x_1(t-\theta_{n1}), x_2(t-\theta_{n2}), \dots, x_n(t-\theta_{nn})\right]. \end{cases} \tag{2}$$

Here each Boolean variable x_i depends on time t and on the state of the other variables x_j in the past. The functions $f_i : \mathbb{B}^n \to \mathbb{B}$, $1 \le i \le n$, are defined via Boolean equations that involve logical operators (see Table 1). Each delay value $\theta_{ij} \in \mathbb{R}$, $1 \le i, j \le n$, is the length of time it takes for a change in variable x_j to affect the variable x_i. One always can normalize delays θ_{ij} to be within the interval (0, 1] so the largest one has actually unit value; this normalization will always be assumed from now on.

Following Dee and Ghil [1], Mullhaupt [2], and Ghil and Mullhaupt [3], we consider in this section only deterministic, *autonomous* systems with no explicit time dependence. Periodic forcing is introduced in Section 3, and random forcing in Section 4. In Sections 2–4 we consider only the case of n finite ("ordinary BDEs"), but in Section 5 we allow n to be infinite, with the variables distributed on a regular lattice ("partial BDEs").