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ScienceDirect
Physica D 237 (2008) 2922–2932
PHYSICA D
www.elsevier.com/locate/physd

Two-component nonlinear Schrödinger models with a double-well potential

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Received 25 September 2007; received in revised form 21 February 2008; accepted 30 April 2008
Available online 15 May 2008

Communicated by A. Pikovsky

Abstract

We introduce a model motivated by studies of Bose–Einstein condensates (BECs) trapped in double-well potentials. We assume that a mixture of two hyperfine states of the same atomic species is loaded in such a trap. The analysis is focused on symmetry-breaking bifurcations in the system, starting at the linear limit and gradually increasing the nonlinearity. Depending on values of the chemical potentials of the two species, we find numerous states, as well as symmetry-breaking bifurcations, in addition to those known in the single-component setting. These branches, which include all relevant stationary solutions of the problem, are predicted analytically by means of a two-mode approximation, and confirmed numerically. For unstable branches, outcomes of the instability development are explored in direct simulations.

Keywords: Symmetry breaking; Bifurcation; Bose–Einstein condensation; Stability

1. Introduction

The nonlinear Schrödinger (NLS) equation is a ubiquitous partial differential equation (PDE) with a broad spectrum of applications, including nonlinear optics in temporal and spatial domains, matter waves, plasmas, water waves, and some biophysical models [1,2]. One of the most fundamental applications of the NLS equation stems from its direct relevance as an accurate mean-field model (known as the Gross–Pitaevskii equation, GPE, in that context) of Bose–Einstein Condensates (BECs) [3,4]. The GPE usually includes an external potential accounting for the magnetic, optical or combined confinement of dilute alkali vapors that constitute the BEC [5]. Basic types of such trapping potentials include parabolic and spatially periodic ones (the latter is created, as an *optical lattice*, by the interference of counter-propagating laser beams). The NLS equations with similar potentials are also relevant models for optical beams in graded-index waveguides and periodic waveguiding arrays [6,7].

A setting that has recently drawn much interest in the context of BECs is based on a double-well potential (DWP), which originates from a combination of the two above-mentioned types of potentials. It was realized experimentally in [8], leading to particularly interesting phenomena including tunneling and Josephson oscillations (for a small number of atoms), or macroscopic quantum self-trapping, with an asymmetric division of atoms between the two wells, for a large number of atoms. Numerous theoretical studies of DWP settings have been performed in parallel with the experimental work [9–17]. They addressed finite-mode reductions, analytical results for specially designed shapes of the potential, quantum effects, and other aspects of the theory (in particular, tunneling between vortex and antivortex states in BEC trapped in a two-dimensional (2D) anisotropic harmonic potential [18] belongs to the latter category).

An interesting generalization of the DWP is provided by multi-well potentials. In particular, nonequilibrium BEC states and generation of quantum entanglement in such settings were recently studied in detail theoretically in [19].

Also considered were 2D [20,21] and 3D extensions of DWP settings, which add one or two extra dimensions to the model, either without any additional potential, or with an independent

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doi:10.1016/j.physd.2008.04.023

optical lattice acting in these directions. The extended geometry makes it possible to consider solitons, self-trapped in the extra dimension(s), which therefore emerge as effectively 1D [20,21] or 2D [22] dual-core states. The solitons may be symmetric, antisymmetric, as well as ones breaking the symmetry between the wells through bifurcations. These states may be described by simplified systems of linearly coupled 1D [20] or 2D [20, 22] PDEs, or, in a more accurate form, effectively 1D solitons can be found as solutions to the full two-dimensional PDE, that takes into regard a particular form of the DWP (which depends on the transverse coordinate and is uniform in the longitudinal direction, allowing the solitons to self-trap in that free direction) [21].

DWPs are also relevant to nonlinear-optics settings, such as twin-core self-guided laser beams in Kerr media [23], optically induced dual-core waveguiding structures in a photorefractive crystal [24], and DWP configurations for trapped light beams in a structured annular core of an optical fiber [25]. As concerns DWPs with extra dimensions, their counterparts in fiber optics are twin–core fibers [26] and dual-core fiber Bragg gratings [27] that were shown to support symmetric and asymmetric solitons. In addition, also investigated was symmetry breaking of solitons in models of twin-core optical waveguides with non-Kerr nonlinear terms, *viz*, quadratic [28] and cubic-quintic (CQ) [29]. All these optical model are based on systems of linearly coupled 1D PDEs. Also in the context of nonlinear optics, a relevant model is based on a system of linearly coupled complex Ginzburg–Landau equations with the CQ nonlinearity, that gives rise to stable dissipative solitons with broken symmetry [30].

In addition to linearly coupled systems of two nonlinear PDEs motivated by the DWP transverse structures, several models of triangular configurations, which admit their own specific modes of symmetry breaking, have also been introduced in optics, each model based on a system of three nonlinear PDEs with symmetric linear couplings between them. These include tri-core nonlinear fibers [31] and fiber Bragg gratings [32], as well as a system of three coupled Ginzburg–Landau equations with intra-core nonlinearity of the CQ type [33].

Significant interest to the DWP settings has also motivated the appearance of rigorous mathematical results regarding symmetry-breaking bifurcations and the stability of nonlinear stationary states [34]. A rigorous treatment was also developed for dynamical evolution in such settings [35,36].

Our objective in this work is to extend the realm of DWPs to multi-component settings. This is of particular relevance to experimental realizations of BEC — e.g., in a mixture of different hyperfine states of ^{87}Rb [3,4] (see also very recent experiments reported in Ref. [37] and references therein). This extension is relevant to optics too, where multi-component dynamics can be, for instance, realized in photorefractive crystals [38]. In the one-component setting, analysis of the weakly nonlinear regime has yielded considerable insight into the emergence of asymmetric branches from symmetric or anti-symmetric ones, in the models with self-focusing and self-defocusing nonlinearity, respectively, and destabilization of the "parent" branches through the respective symmetry-breaking bifurcations, as well as the dynamics initiated by the destabilization [9,10,13,15,16,34].

In the present work, we extend analysis of the DWP setting to two-component systems. In addition to the finite-mode approximation, which can be justified rigorously [34] under conditions that remain valid in the present case, we use numerical methods to follow the parametric evolution of solution branches that emerge from bifurcations in the two-component system. We observe that, in addition to the branches inherited from the single-component model, new branches appear with the growth of nonlinearity. We dub these new solutions "combined" ones, as they involve both components. In fact, the new branches connect some of the single-component ones. Furthermore, these new branches may undergo their own symmetry-breaking bifurcations. The solutions depend on chemical potentials of the two species, μ_1 and μ_2 (in terms of BEC), and, accordingly, loci of various bifurcations will be identified in the plane of (μ_1, μ_2).

The paper is structured as follows. In Section 2, we present a framework of the two-component problem, including the two-mode reduction (in each component), that allows us to considerably simplify the existence and stability problems. The formulation includes also a physically possible linear coupling between components, but the actual analysis is performed without linear coupling. In Section 3, we report numerical results concerning the existence and stability of the new states, as well as the evolution of unstable ones. In Section 4, we summarize the findings and discuss directions for further studies.

2. Analytical consideration

As a prototypical model that is relevant both to BECs [3, 4] and optics [6], we take the normalized two-component NLS equation of the following form:

$$\begin{aligned} \mathrm{i}u_t &= Lu + \kappa v + \sigma(|u|^2 + g|v|^2)u - \mu_1 u, \\ \mathrm{i}v_t &= Lv + \kappa u + \sigma(|v|^2 + g|u|^2)v - \mu_2 v, \end{aligned} \tag{1}$$

where u and v are the wave functions of the two BEC components, or local amplitudes of the two optical modes, $\mu_{1,2}$ are chemical potentials in BEC or propagation constants in the optical setting, κ is the coefficient of the linear coupling between the components, and

$$L = -(1/2)\partial_x^2 + V(x) \tag{2}$$

is the usual single-particle operator with trapping potential $V(x)$. The repulsive or attractive character of the nonlinearity in BEC (self-defocusing or self-focusing, in terms of optics) is defined by $\sigma = +1$ and $\sigma = -1$, respectively, while σg is the coefficient of inter-species interactions in BEC, or cross-phase modulation (XPM) in optics.

In the case of (for instance) two hyperfine states with $|F, m_F\rangle = |1, -1\rangle$ and $|2, 1\rangle$ in ^{87}Rb, both coefficients of the self-interaction and the cross-interaction coefficient are very close, although their slight difference is critical in accounting