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Synchronization in complex dynamical networks with nonsymmetric coupling

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Abstract

Based on the work of Nishikawa and Motter, who have extended the well-known master stability framework to include non-diagonalizable cases, we develop another extension of the master stability framework to obtain criteria for global synchronization. Several criteria for global synchronization are provided which generalize some previous results. The Jordan canonical transformation method is used in stead of the matrix diagonalization method. Especially, we show clearly that, the synchronizability of a dynamical network with nonsymmetric coupling is not always characterized by its second-largest eigenvalue, even though all the eigenvalues of the nonsymmetric coupling matrix are real. Furthermore, the effects of the asymmetry of coupling on synchronizability of networks with different structures are analyzed. Numerical simulations are also done to illustrate and verify the theoretical results on networks in which each node is a dynamical limit cycle oscillator consisting of a two-cell cellular neural network.

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Keywords: Complex network; Global synchronization; Exponential stability; Jordan canonical transformation

1. Introduction

Complex networks have become a focal research topic of the nonlinear dynamics community in the past decade, especially how the topological structure affects the dynamics of networks [1–9]. One of the most remarkable phenomena in the dynamics of networks is their spontaneous synchronization, which has been carefully studied in recent years [10–36]. Barahona and Pecora showed how the addition of random shortcuts translates into improved network synchronizability [11]. Wang and Chen presented a uniform complex network model [9], and investigated its synchronization in small-world and scale-free networks [12–15]. Nishikawa et al. found that networks with a homogeneous distribution of connectivity are more synchronizable than heterogeneous ones, even though the average network distance is larger [16]. Chen proposed some criteria based on the concept of matrix measure [17]. Zhou et al. studied synchronization in weighted complex networks and showed that the synchronizability of random networks with a large minimum degree is determined by two leading parameters: the mean degree and the heterogeneity of the distribution of node's intensity [18]. Based on delayed complex network models, some synchronization criteria for both delay-independent and delay-dependent exponential stability of the synchronization state are derived [19–21]. Lü and Chen introduced a general time-varying complex network model and derived some synchronization criteria for time-varying complex networks [22,23]. While, most of the above researches follow from the general master stability framework proposed by Pecora and Carroll [24], which is based on the diagonalization of its variational equation at the synchronous state, thus require the coupling matrix to be symmetric or diagonalizable and usually gives criteria for local synchronization. Whereas, many realistic networks are nonsymmetric, and maximally synchronizable networks are almost always nondiagonalizable and can

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be obtained by imposing unidirectional information flow with normalized input strength [25]; for a given degree distribution, the maximum synchronizability is achieved when the network of couplings is weighted and directed and the overall cost involved in the couplings is minimum [26]. Nishikawa and Motter extended the master stability framework to include cases where the diagonalization is not necessarily possible by using the Jordan canonical form [25,27], which has been used to study the stability of synchronization in nonsymmetric networked systems [28–31]. We say nonsymmetric network which means that its coupling matrix is not constrained to be symmetric. Very recently, we used Jordan canonical transformation to observer design of nonsymmetric networks [32] and synchronization analysis of nonsymmetric networks with coupling time-delay [33].

In this paper, based on the work of Nishikawa and Motter [25,27], we extend the general master stability framework [24] from local synchronization to global synchronization by using an assumption which is satisfied for many existing chaotic (or limit cycle) oscillators; the Jordan canonical transformation method is used instead of the matrix diagonalization method. Some criteria for global synchronization are derived, in which the Jacobian matrix of the dynamical node evaluated on the synchronous state is not necessarily used. Meanwhile, the effects of the asymmetry of coupling on synchronizability of networks with different topological structure are analyzed. We noted that some criteria for global synchronization in networks have been presented in [28–31]; Li and Chen have derived a sufficient condition for the global synchronization and asymptotical stability by introducing a reference state with the Lyapunov stability theorem [34].

This paper is organized as follows. In Section 2, the dynamical complex network model considered is introduced, and some mathematical definitions and lemmas are given. The main results of this paper are given in Section 3, in which we extend the master stability framework for global synchronization, and some global synchronization criteria are obtained based on the general network model with nonsymmetric coupling. In Section 4, the effects of the asymmetry of coupling on the synchronizability of networks with different topological structures are analyzed. Numerical simulations are given in Section 5. Section 6 is the conclusion of this paper.

Now, we list some mathematical notations used in this paper. We denote matrix $\boldsymbol{A}$ is an $n \times n$ complex (real) matrix by $A \in M_n(C)$ ($A \in M_n(R)$). A square matrix $\boldsymbol{A}$ is positive definite if $\bar{\eta}(t)^* A\bar{\eta} > 0$ for all $\bar{\eta} \neq 0$, and a square matrix $\boldsymbol{A}$ is negative definite if $\bar{\eta}(t)^* A\bar{\eta} < 0$ for all $\bar{\eta} \neq 0$. We denote the positive (nonnegative) definiteness of $\boldsymbol{A}$ by $A \succ 0 (A \succeq 0)$ and the negative (nonpositive) definiteness of A by $A \prec 0 (A \preceq 0)$. $A \succ B$ means $A - B$ is a positive definite matrix. The vector norm used will be $\|x\| = (x^* x)^{\frac{1}{2}}$. $\mathrm{Re}(\lambda_i)$ and $\mathrm{Im}(\lambda_i)$ denote the real part and imaginary part of a complex number λ_i, respectively. $w_l^R = \mathrm{Re}(w_l(t))$, $w_l^I = \mathrm{Im}(w_l(t))$, $\dot{w}_l^R(t) = \frac{\mathrm{d}w_l^R(t)}{\mathrm{d}t}$, and $\dot{w}_l^I(t) = \frac{\mathrm{d}w_l^I(t)}{\mathrm{d}t}$, where $w_l(t)$ is a complex variable. The symmetric part of a square matrix is $A^s = \frac{1}{2}(A^{\mathrm{T}} + A)$, where $A \in M_n(R)$. The $n \times n$ identity matrix is denoted by I_n.

2. Network model and mathematic preliminaries

2.1. Network model

Lü et al. introduced a general dynamical network model [35]. The state equations of the entire network are given by

$$\dot{x}_i(t) = f(x_i(t)) + \sum_{j=1}^{N} g_{ij} A x_j(t), \quad i = 1, \ldots, N, \tag{1}$$

where $x_i(t) = (x_{i1}(t), x_{i2}(t), \ldots, x_{in}(t))^{\mathrm{T}} \in R^n$ is the state variable of node i. $A = (a_{ij}) \in M_n(R)$ is a constant inner-coupling matrix between nodes, and $G = (g_{ij}) \in M_N(R)$ is the coupling matrix of the network, where $g_{ij} \in R$ is defined as follows: if there is a connection from node j to node i ($i \neq j$), then, the coupling strength $g_{ij} \neq 0$; otherwise, $g_{ij} = 0$, and the diagonal elements of G are defined by

$$g_{ii} = -\sum_{\substack{j=1 \\ j\neq i}}^{N} g_{ij}, \quad i = 1, \ldots, N. \tag{2}$$

In network model (1), the coupling matrix G is nonsymmetric, and its off-diagonal elements may be negative.

The coupling matrix G may have complex eigenvalues; even if all its eigenvalues are real, in general, it is not diagonalizable [36]. We always assume that G is irreducible and all of its off-diagonal elements are nonnegative. According to Lemma 2 in Ref. [37], the real parts of the eigenvalues of G are less than or equal to 0, and zero is an eigenvalue of G with multiplicity one corresponding to eigenvector $(1, 1, \ldots, 1)^{\mathrm{T}}$. For simplicity, we always assume $\lambda_1 = 0$. If the coupling matrix has k different eigenvalues and all of them are taken to be real, we denote them as $0 = \lambda_1 > \lambda_2 > \lambda_3 > \cdots > \lambda_k$.

Network (1) can be equivalently expressed as (3) by using the Kronecker product [24,37–39],

$$\dot{X}(t) = f(X(t)) + (G \otimes A)X(t) \tag{3}$$

where $X(t) = (x_1(t), x_2(t), \ldots, x_N(t))^{\mathrm{T}} \in R^{nN}$, $\dot{X}(t) = (\dot{x}_1(t), \dot{x}_2(t), \ldots, \dot{x}_N(t))^{\mathrm{T}} \in R^{nN}$, $f(X(t)) = (f(x_1(t)), f(x_2(t)), \ldots, f(x_N(t)))^{\mathrm{T}} \in R^{nN}$, $G \otimes A \in M_{nN}(R)$ denotes the Kronecker product of matrices G and A. It is sometimes more convenient to use (3) than (1).

2.2. Mathematic preliminaries

Definition 1 (*Synchronization*). Let D^0 denote an open set in the state space Ω, if from any initial point $X(t_0) = (x_1(t_0), \ldots, x_i(t_0), \ldots, x_N(t_0))^{\mathrm{T}} \in D^0$, there is $\|x_i(t) - s(t)\| \to 0$ as $t \to \infty$, $i = 1, \ldots, N$, the autonomous network (1) is said to realize synchronization locally in D^0, and $S(t) = (s(t), \ldots, s(t))^{\mathrm{T}}$ is called the synchronous state. If D^0 is the entire state space Ω of network (1), then it is said to realize synchronization globally.

Obviously, if network (1) realized synchronization, then $x_1(t) = \cdots = x_N(t) = s(t)$, $s(t)$ is the synchronous state