Physica D 238 (2009) 1345–1351

Contents lists available at ScienceDirect

Physica D

journal homepage: www.elsevier.com/locate/physd

Modulational instability of matter waves under strong nonlinearity management

F.Kh. Abdullaev *, A.A. Abdumalikov, R.M. Galimzyanov

Physical-Technical Institute of the Uzbek Academy of Sciences, 2-b, G. Mavlyanov street, 100084, Tashkent, Uzbekistan

ARTICLE INFO

Article history:
Available online 19 November 2008

PACS:
03.75.Lm
03.75.-b
30.Jp

Keywords:
Modulational instability
Matter wave
Feshbach resonance management
Optical lattice
Gap soliton

ABSTRACT

We study modulational instability of matter-waves in Bose–Einstein condensates (BEC) under strong temporal nonlinearity-management. Both BEC in an optical lattice and homogeneous BEC are considered in the framework of the Gross–Pitaevskii equation, averaged over rapid time modulations. For a BEC in an optical lattice, it is shown that the loop formed on a dispersion curve undergoes transformation due to the nonlinearity-management. A critical strength for the nonlinearity-management strength is obtained that changes the character of instability of an attractive condensate. MI is shown to occur below (above) the threshold for the positive (negative) effective mass. The enhancement of number of atoms in the nonlinearity-managed gap soliton is revealed.

1. Introduction

The phenomenon of modulational instability (MI) of nonlinear plane waves under different types of management of the system parameters has been the subject of intensive research over the last years [1]. Main emphasis was given to dispersion-management and nonlinearity-management. In nonlinear optics strong and rapid modulations of the fiber dispersion is achieved by periodic arrangement of fiber spans with alternating sign of the dispersion. Dispersion-managed solitons supported by such a system have essential advantages over conventional optical solitons for long distance communication purposes [2–4]. Modulations of the nonlinearity is a challenging problem also in fiber ring lasers and in generation of Faraday waves in Bose–Einstein condensates (BEC) [5–9]. MI in the form of Faraday waves can be observed both in *attractive* and *repulsive* condensates. Recent observation of the MI in optical media resulted from the periodic modulation of the nonlinearity in the evolution variable, confirms the existence of parametric resonances in the MI growth rate [6,10,11]. Faraday waves (parametrically excited waves) in a BEC emerging from temporal periodic variation of the atomic scattering length have been studied in [9]. Such type of modulations can be achieved by variation of the external magnetic field near Feshbach resonances (FR). The corresponding technique is known as FR management. In the Gross–Pitaevskii equation this corresponds to a temporal variation of the mean-field nonlinearity, i.e. to the nonlinearity-management. MI in a harmonically trapped BEC under FR management has been investigated in [12].

Recently the strong dispersion-management has been applied to the dynamics of nonlinear periodic waves, namely cnoidal waves, in optical fibers [13,14]. In these works the existence of dispersion-managed cnoidal waves and strong deviation of the stability borders of these waves from the ones of standard cnoidal wave solutions of the nonlinear Schrödinger equation (NLSE) have been established. Extension of the stability regions of some types of nonlinear periodic waves can be due to the different scenarios for the onset of MI of the background plane waves. Adiabatic FR management for cnoidal waves in optical lattices has been considered in [15,16]. The case of strong nonlinearity-management remains unexplored.

The strong nonlinearity-management may be an effective tool for stabilization of matter-wave solitons in multi-dimensional attractive BEC [17–26]. In the context of nonlinear optics such stabilization mechanism was first discussed in [27,28]. The phenomenon of MI is particularly important for generation of soliton trains in BEC with controlled spatial arrangement (repetition rate). MI of BEC in linear and nonlinear optical lattices in the absence of time-periodic nonlinearity-management has been investigated in our recent work [29]. Here we consider both the MI of a homogeneous BEC and MI of a BEC loaded in an optical lattice under FR management. The gap soliton structure existing in a BEC with the zero background scattering length ($a_{sb} = 0$) has been investigated in Ref. [30]. The couple-mode theory can be used to analyze MI of nonlinear plane waves in an optical lattice subject to FR management. In our investigations particular interest will be

* Corresponding author. Fax: +998 712 35 42 91.
E-mail address: fatkh@uzsci.net (F.Kh. Abdullaev).

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doi:10.1016/j.physd.2008.11.001

paid to the properties of loop structures emerging in the band gaps (forbidden band).

In the present paper we investigate nonlinear dispersion relations and the process of MI in a BEC under strong temporal nonlinearity management (SNM). The outline of the paper is as follows: The mathematical model is formulated in Section 2; MI in a homogeneous BEC under SNM is considered in Section 3; The nonlinear dispersion relation and loop structures for BEC in an optical lattice under SNM are analyzed in Section 4 using the coupled-mode theory. This section also includes the regions of MI found in different areas of the band structure; The properties of gap solitons are investigated in Section 5; Section 6 is devoted to details of our numerical procedure; In the final Section 7 we summarize our main results.

2. The model

Let us consider a BEC under temporal Feshbach resonance management when the scattering length a_s varies in time. Then an elongated BEC can be described by the quasi-1D GP equation with a periodic potential (optical lattice) and the time-dependent management of the coefficient of nonlinearity

$$i\hbar\psi_t = -\frac{\hbar^2}{2m}\psi_{xx} + V(x)\psi - g_{1D}(t)|\psi|^2\psi, \quad (1)$$

where $g_{1D}(t) = 2\hbar a_s(t)\omega_\perp$ is the mean field nonlinearity coefficient, $\omega_\perp$ is the transverse oscillator frequency and $V(x) = V_0\cos^2(kx)$ is an optical lattice potential, $\int_{-\infty}^{\infty} dx|\psi|^2 = N$, N is the number of atoms. In dimensionless units we have

$$x \to kx, \qquad t \to \omega_R t, \qquad \epsilon = \frac{V_0}{2E_R}, \qquad E_R = \frac{\hbar^2 k^2}{2m},$$
$$\omega_R = E_R/\hbar, \qquad u = \sqrt{\frac{2\hbar a_s\omega_\perp}{E_R}}\,\psi e^{-i\epsilon t}.$$

Eq. (1) takes the form of the NLSE with varying in time mean field nonlinearity coefficient

$$iu_t + u_{xx} + \gamma(t)|u|^2u - 2\epsilon\cos(2x)u = 0, \quad (2)$$

where $\gamma(t)$ describes the strong nonlinearity-management and has the form

$$\gamma(t) = \gamma_0 + \frac{1}{\mu}\gamma_1\left(\frac{t}{\mu}\right), \qquad \int_0^1 \gamma_1(\tau)\,d\tau = 0, \qquad \tau = \frac{t}{\mu}, \quad \mu \ll 1. \quad (3)$$

This model has been considered in recent papers [15,16,30]. Specifically, in works [15,16] the evolution of nonlinear periodic waves under adiabatic time-variation of the scattering length has been studied and a possibility of generation of a train of solitons by such a management scheme has been shown. Properties of gap solitons under the strong management of nonlinearity were analyzed based on the coupled mode system of equations in [30]. In this work the gap soliton solutions and their stability for the case $\gamma_0 = 0$ were investigated. Here we will study MI of nonlinear plane waves in a BEC (without, and with an optical lattice) under SNM, as well as properties of gap solitons in the model (2) for nonzero value of γ_0. In particular, we will analyze the possibility of enhancement of number of atoms in the gap soliton under SNM.

In deriving averaged equation we follow the works [30,31] and use the transformation

$$u(x,t) = e^{i\gamma_{-1}(t)|v|^2}v(x,t),$$
$$\gamma_{-1}(\tau) = \int_0^1 \gamma(\tau')\,d\tau' - \int_0^1\int_0^\tau \gamma(\tau')\,d\tau'\,d\tau. \quad (4)$$

Supposing the parameter μ to be small (that corresponds to high frequencies of modulation) unknown function v can be expanded in series as

$$v = w + \mu v_1 + \mu^2 v_2 + \cdots, \quad (5)$$

where unknown w is a slowly varying function. Using transformation (4) and expansion (5) in governing Eq. (2) with posterior averaging over the period of rapid modulation, we arrive at the following averaged equation for w [31]

$$iw_t + w_{xx} + \gamma_0|w|^2w - 2\epsilon\cos(2x)w + \sigma^2[2(|w|^2)_{xx}|w|^2 + ((|w|^2)_x)^2]w = 0. \quad (6)$$

Parameter σ is defined as $\sigma^2 = \int_0^1 \gamma_{-1}^2\,d\tau$. For particular case of sinusoidal modulations $\gamma_1 = h\sin(\omega t)$ we have $\sigma^2 = h^2/(2\omega^2) \sim O(1)$ $(\omega = 1/\mu)$. For the step-like modulation with the same amplitude h and frequency ω we have $\sigma^2 = h^2/\omega^2$.

This form of averaged equation can be also obtained for the case of the weak nonlinearity management when $\gamma = \gamma_0 + \gamma_1(t/\mu)$, with $\sigma^2 \ll 1$ [31,32].

3. Modulational instability of nonlinear plane wave in a homogeneous media

Now let us consider the case when the optical lattice is switched off, i.e. $\epsilon = 0$ in Eq. (2). The MI of a nonlinear plane wave $w = A\exp(i(\gamma_0 A^2 t))$ can be explored using the linear stability analysis, i.e. looking for the solution in the form

$$w = (A + \psi(x,t))\exp[i\gamma_0 A^2 t], \quad \psi \ll A. \quad (7)$$

We have the following equation for ψ

$$i\psi_t + \psi_{xx} + \gamma_0 A^2(\psi + \psi^*) + 2\sigma^2 A^4(\psi_{xx} + \psi_{xx}^*) = 0. \quad (8)$$

Representing $\psi = \psi_r + i\psi_i$ and performing Fourier transformation $\psi_r(\psi_i)(x,t) = \int dk\bar{u}(\bar{v})(k,t)\exp(ikx)$ we get the dispersion relation

$$p^2 = k^2[2\gamma_0 A^2 - (1 + 4\sigma^2 A^4)k^2]. \quad (9)$$

Instability region corresponds to the condition $p^2 > 0$. Thus we obtain

$$k^2 \le \frac{2\gamma_0 A^2}{1 + 4\sigma^2 A^4}. \quad (10)$$

The maximum of the MI gain is achieved at the value of the wave number

$$k_c = \sqrt{\frac{\gamma_0}{1 + 4\sigma^2 A^4}}A. \quad (11)$$

Maximal value of the MI growth rate is

$$p_c = \frac{\gamma_0 A^2}{\sqrt{1 + 4\sigma^2 A^4}}. \quad (12)$$

Thus we find that under the temporal nonlinearity management the MI growth rate is decreased by a factor of $\sqrt{1 + 4\sigma^2 A^4}$. Such decrease of the gain is due to the defocusing effect induced by the nonlinearity management. This observation explains the stabilizing role of the strong nonlinearity management in a higher dimensional attractive BEC [17,18,33,34].

Numerical simulations of the 1D GP Eq. (2) with a strong nonlinearity management confirm these predictions. In Fig. 1 we plot the MI gain versus the wave number of modulations k for three different cases with $\gamma_0 = 1$ and $\omega = 10$: (a) when the nonlinearity-management is absent, $\sigma^2 = 0$ and when the management is present (b) $\sigma^2 = 0.125(h = 5)$, (c) $\sigma^2 = 0.5(h = 10)$. One