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Matter-wave vortices and solitons in anisotropic optical lattices

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ABSTRACT

Using numerical methods, we construct families of vortical, quadrupole, and fundamental solitons in a two-dimensional (2D) nonlinear-Schrödinger/Gross–Pitaevskii equation which models Bose–Einstein condensates (BECs) or photonic crystals. The equation includes the attractive or repulsive cubic nonlinearity and an anisotropic periodic potential. Two types of anisotropy are considered, accounted for by the difference in the strengths of the 1D sublattices, or by a difference in their periods. The limit case of the quasi-1D optical lattice (OL), when one sublattice is missing, is included too. By means of systematic simulations, we identify stability limits for two species of vortex solitons and quadrupoles, of the rhombus and square types. In the attraction model, rhombic vortices and quadrupoles remain stable up to the limit case of the quasi-1D lattice. In the same model, finite stability limits are found for vortices and quadrupoles of the square type, in terms of the anisotropy parameter. In the repulsion model, rhombic vortices and quadrupoles are stable in large parts of the first finite bandgap (FBG). Another species of partly stable anisotropic states is found in the second FBG, *subfundamental dipoles*, each squeezed into a single cell of the OL. Square-shaped quadrupoles are completely unstable in the repulsion model, while vortices of the same type are stable only in weakly anisotropic OL potentials.

1. Introduction

A subject of great current interest in theoretical and experimental studies of the dynamical patterns formation in Bose–Einstein condensates (BECs) and photonic crystals is the existence and stability of multidimensional solitons and localized vortices in two- and three-dimensional (2D and 3D) settings. While solitons have been created in well-known experiments in BEC with attractive interactions between atoms, loaded in effectively 1D (cigar-shaped) traps [1,2], a basic problem impeding straightforward creation of multidimensional solitons is the instability against collapse driven by the cubic self-attractive nonlinearity [3]. As concerns ring-shaped solitons with intrinsic vorticity, they are additionally subject to the symmetry-breaking azimuthal instability, which is even stronger than the instability to the collapse [4].

It was proposed [5–8] that a general method for the stabilization of multidimensional solitons and localized vortices in BEC may be based on the use of optical lattices (OLs), i.e., periodic potentials which are induced by the interference between counterpropagating laser beams illuminating the condensate. Stable 2D and 3D solitons can be supported by *full* OLs (with dimension D equal to that of the space in which the lattice was created), and also by *low-dimensional* OLs, with dimension $D-1$, i.e., quasi-1D and quasi-2D lattices in the 2D and 3D space, respectively [7,10]. In the latter case, the solitons naturally feature a strongly anisotropic shape.

Although the OL breaks the rotational invariance, it can support and stabilize not only fundamental solitons, but also vortical ones [5,6,8,9], including localized vortices of higher orders, with "spin" (the topological charge, alias vorticity) $S \geq 1$ [11]. The simplest "crater-shaped" vortex soliton, in the form of a single density peak with an inner hole induced by the vorticity, is always unstable [12,13] (the instability splits it into several pulses, one of which survives, demonstrating a random walk across the OL [13]). Stable vortices with $S = 1$ are built as sets of four [5,6] or eight [12,23] peaks, with the phase difference, respectively, $\Delta\phi = \pi/2$ or $\pi/4$ between adjacent ones, which corresponds to the total phase circulation of 2π. There are two different species of the simplest stable vortex solitons, which are composed of four peaks: densely packed *squares*, with the center coinciding with a local maximum of the OL potential [6,13], and "porous" *rhombuses*, featuring a nearly empty lattice cell at the center [5,12,13] (two similar species of vortex solitons are also known in discrete models, where rhombuses are sometimes called *crosses* [14]). It was recently

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demonstrated [9] that the squares and rhombuses form separate families, each featuring a pair of branches connected at a turning point. The modes belonging to the different branches differ by the number of peaks (for instance, four and eight peaks that may constitute a rhombic vortex, as mentioned above).

Stable vortices of higher orders, up to $S = 6$, were constructed as ring-shaped sets consisting of up to $N = 12$ peaks, with respective phase differences $\Delta\phi = 2\pi S/N$ [11]. The localized vortices formally corresponding to $S = 2$, which are composed of 4 peaks, have $\Delta\phi = \pi$, i.e., these are actually real solutions in the form of *quadrupoles*. Also predicted were stable topological patterns in the form of *supervortices*, which are ring chains built of 12 (or more) compact crater-shaped vortices carrying local spins $s = 1$, with global vorticity $S = \pm 1$ imprinted onto the entire ring [11]. The supervortices may be stable, even if individual "crater" peaks, of which they are composed, are unstable in isolation, as mentioned above.

The quasi-2D lattice in the 3D space can also support stable 3D solitons with embedded vorticity [15]. On the other hand, the existence and stability of vortices in the 2D model with the quasi-1D lattice remains an open problem.

Solitons of a different type, namely, *gap solitons* (GSs), can be supported by full OL potentials of any dimension in BEC with repulsive interactions between atoms. The GSs result from the balance between the repulsive nonlinearity and the negative effective mass in parts of the linear bandgap spectrum generated by the OL [16,17]. They are stable localized objects [18], even if they cannot realize the ground state of the condensate trapped in the OL. The creation of GSs containing $\simeq 250$ atoms of ^{87}Rb was reported in the effectively 1D setting [19] (see also review [20]). Multidimensional GSs [21], including gap-type vortices [12,22, 23], and "semi-gap" solitons (which are organized as GSs in one direction, and regular solitons in the other [24]) were predicted too. As concerns gap-soliton vortices, they may feature both the square [22] and rhombic [12,22,23] shapes. In both 1D and 2D settings, stable GSs may be supported not only by periodic OLs, but also by quasi-periodic lattices [25].

Thus far, the studies of 2D and 3D solitons supported by OLs were confined to two limit cases, *viz.*, the full (isotropic) lattices, and low-dimensional ones, with one sublattice missing. In experiments, it is quite easy to create a more general setting, with an *anisotropic* OL, composed of 1D sublattices with different strengths and/or different periods. To the best of our knowledge, fundamental solitons and vortices in anisotropic lattices were previously studied only in the discrete model [14], that may be considered as a model for the BEC trapped in a very deep OL [17,26].

Similar settings are available for the experiment in nonlinear optics, where photonic lattices can be induced in photorefractive crystals illuminated by pump laser beams in the ordinary polarization (in which the medium is nearly linear), while solitons are created by probe beams launched in the extraordinary polarization [27]. In addition to fundamental 2D solitons [28], localized vortices [29], necklace-shaped [30] and circular [31] solitons have been created by means of this technique. In particular, 2D anisotropic solitons supported by an isotropic square-shaped photo-induced lattice were reported in Ref. [34]. Asymmetrically shaped vortex solitons were predicted in that medium too [35].

Another realization of 2D solitons [32] and vortices [33] in nonlinear optics is possible in photonic-crystal fibers. Unlike the saturable nonlinearity characteristic to photorefractive crystals, they feature the same cubic (Kerr) nonlinearity as BEC.

The anisotropic-lattice settings for solitons, vortices and quadrupoles, which are the subject of the present work, suggest to consider several issues of evident interest. One of them is finding stability borders for 2D vortex and quadrupole solitons in the model with the attractive cubic nonlinearity. Another straightforward question is to identify existence and stability limits for 2D GSs in the model with the self-repulsion, where, obviously, solitons cannot exist in the quasi-1D limit, when one of 1D sublattices is switched off. The first noteworthy finding reported below is that both the vortices (with $S = 1$) and quadrupoles of the rhombus type exist and *remain stable* up to the limit of the quasi-1D lattice (in the attraction model). For the square-shaped vortices and quadrupoles, we find a critical degree of the anisotropy of the 2D lattice, up to which they remain stable – unlike their rhombic counterparts, they are unstable in the limit of the quasi-1D lattice. Generally, the stability region for quadrupoles in the attraction model is essentially broader than for vortices. In the repulsion model, we find the existence and stability limits for the fundamental ($S = 0$), vortical ($S = 1$) and quadrupole GSs. In this case too, the rhombuses are much more stable than squares, but quadrupoles are found to be *less stable* than vortices, on the contrary to the attraction model.

These results predict universal properties of fundamental, vortical, and quadrupole 2D solitons in nonlinear periodic media, that can be realized in BEC, photonic crystals and photonic-crystal fibers, and other physical media. Experimental verification of the predictions is quite feasible both in BEC and photonic lattices in photorefractive crystals.

The rest of the paper is organized as follows. In Section 2, we formulate the model and demonstrate spectra generated by the OLs in its linear version. Systematic findings for the 2D vortices and quadrupoles of the rhombus and square type are reported in Sections 3 and 4, for the models with attraction and repulsion, respectively. While the results are obtained by means of systematic simulations, Section 3 includes a brief discussion which aims to explain some findings by means of an analytical approximation. The paper is concluded by Section 5 in which the main results are summarized, and examples of stable three-dimensional GSs supported by the respective anisotropic OL are additionally displayed.

2. The model: nonlinear equations and linear spectra

2.1. The Gross–Pitaevskii equation

The starting point is the 3D Gross–Pitaevskii equation for the mean-field wave function, $\Psi(X, Y, Z, T)$, where the coordinates and time denoted by capital letters are measured in physical units:

$$i\hbar\frac{\partial\Psi}{\partial T} = -\frac{\hbar^2}{2m}\left(\frac{\partial^2\Psi}{\partial X^2}+\frac{\partial^2\Psi}{\partial Y^2}+\frac{\partial^2\Psi}{\partial Z^2}\right) + \frac{4\pi\hbar^2 a_s}{m}|\Psi|^2\Psi + W(X,Y,Z)\Psi. \quad (1)$$

Here m and a_s are the atomic mass and scattering length of atomic collisions, and W is the external potential. For the *anisotropic* 3D lattice with strength W_0 and period d, the potential is

$$W = -W_0\left[\eta\cos\left(\frac{2\pi X}{d}\right)+\cos\left(\frac{2\pi Y}{d}\right)+\cos\left(\frac{2\pi Z}{d}\right)\right], \quad (2)$$

where anisotropy factor η takes values $0 < \eta \leq 1$, with the isotropic and quasi-2D limits corresponding, respectively, to $\eta = 1$ and $\eta = 0$. If the 1D components of the OL are induced by the superposition of two counterpropagating laser beams with wavelength λ and misalignment angle 2θ, the corresponding OL period is $d = \lambda/(2\cos\theta)$. Besides using different intensities of light in different pairs of beams, which is accounted for by $\eta < 1$ in Eq. (2), another source of the anisotropy may be the use of different