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Condensation of classical nonlinear waves in a two-component system

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ABSTRACT

We study the formation of large-scale coherent structures (condensates) for a system of two weakly interacting Bose gases in the semiclassical approximation. Using the coupled defocusing nonlinear Schrödinger (NLS) equations as a representative model, we focus on condensation in the phase mixing regime. We employ weak turbulence theory to provide a complete thermodynamic description of the classical condensation process. We show that the temperature and the condensate mass fractions are fully determined by the total number of particles in each component and the initial total energy. Moreover, we find that, at higher energies, condensation can occur in only one component. We derive an analytic result for the variation of the critical energy where this transition occurs. The theory presented provides excellent agreement with results of numerical simulations obtained by directly integrating the dynamical model.

1. Introduction

Many classical systems in nature reveal the emergence of large scale coherent structures from a background irregular field characterised by small-scale fluctuations. Examples of systems that exhibit such behaviour include classical turbulence, nonlinear optics, superfluids, ultracold gases and Bose–Einstein Condensates (BECs), and the formation of the early universe. In certain regions of the parameter space, a large sub-class of these systems can be described by a system of weakly nonlinear dispersive waves. A universal equation that governs the evolving field in such scenarios is then given by the Nonlinear Schrödinger (NLS) equation. The process of self-organisation in the focusing NLS equation has been studied in [1]. It was found that a large-scale solitary wave tends to emerge from a sea of small-scale turbulent fluctuations. In this work, we concentrate on the defocusing NLS equations. This equation has been receiving increasing attention due to the experimental advances in BECs. In this context, the defocusing NLS equation corresponds to the Gross–Pitaevskii (GP) equation [2] of a homogeneous Bose gas. The GP equation has long been used as a model of a weakly interacting Bose gas at zero temperature. More recently, it has been argued [3] that the GP equation can be used to model the long wavelength part of the spectrum of a BEC at finite temperatures. Numerical simulations conducted within this framework [4,5], for a one component system, have indeed confirmed this, revealing the ability of the model to capture the formation of a large scale coherent structure (a condensate) from an initially turbulent state. More recently, Connaughton et al. [6] formulated a theory, for the one component system, which provides a framework for determining the state of the system at equilibrium. This in turn can be used to compute the thermodynamic properties of a weakly interacting Bose gas.

With the rapid developments being made in experimental techniques, it is now possible to realise multicomponent BECs formed by the simultaneous trapping and cooling of atoms in distinct spin or hyperfine levels [7] or different atomic species [8]. Such systems are of interest since they can give rise to phenomena which are unique to multicomponent systems that have no analogue in the one component case. The finite temperature dynamics of such Bose gas mixtures is governed by a system of coupled NLS equations. An important question that subsequently arises is "*How can we describe the thermodynamic state of such a multicomponent system?*" This would then allow a clear specification of the temperature of the system which in turn would provide a means to quantify various dynamical effects. As an example, we might be interested in determining the mutual friction forces that can arise between the thermal cloud and superfluid vortices in a multicomponent system. Results addressing these issues for the one component case have already been presented in [9]. To generalise these results to a multicomponent system, an accurate specification of the properties of the system at equilibrium is needed.

In this work, we will take the first step towards generalising the results presented in [6], for a one component system, to an N-component system. Since the analysis of the N-component case is technically more involved but the ideas are in principle similar to the simpler two component case, we will focus on this special case in the remainder of this work. We note that

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even this simpler case can serve as an idealised model in various branches of physics. For example, the model can be used to study symmetry-breaking phase transitions that are believed to have occurred in the early evolution of the universe. A specific example is given by the Kibble–Zurek mechanism [10] of the formation of topological defects following the rapid quench of the system below the point of second-order phase transitions. This scenario would correspond to the formation of cosmological *vortons* and *springs* that are analogous to the vortex ring-slaved wave and vortex ring-vortex ring complexes of BECs [11]. In addition to these physical examples, the two-component coupled NLS equations are also encountered in the study of optical fibres and electromagnetic waves [12]. Given the universality of these equations in the nonlinear sciences, an accurate thermodynamic description of the condensation process in such a system could find widespread applications in the aforementioned fields. We recall that two-component systems tend to show a broad class of qualitatively different behaviour depending on the relative strengths of the intercomponent and intracomponent interactions. This can lead to contrasting regimes of condensation: the phase mixing regime and the phase separation regime [13]. Consistent with the assumptions commonly used in weak turbulence theory, which will form the basis of our approach, we will focus exclusively on the phase mixing regime.

2. Kinetic theory for two-component system

We begin by considering the scenario of a system of two weakly interacting Bose gases within the semiclassical approximation, that have been rapidly cooled below the transition temperature. Their evolution from the resulting strongly nonequilibrium initial state is then described by the coupled NLS equations given by

$$\begin{aligned} \mathrm{i}\partial_t\psi_1 &= -\nabla^2\psi_1 + |\psi_1|^2\psi_1 + \alpha|\psi_2|^2\psi_1, \\ \mathrm{i}\partial_t\psi_2 &= -\nabla^2\psi_2 + |\psi_2|^2\psi_2 + \alpha|\psi_1|^2\psi_2, \end{aligned} \tag{1}$$

where ψ_1 and ψ_2 are complex-valued classical fields corresponding to each component, and α is the intracomponent coupling constant. For the phase mixing regime and repulsive interactions between atomic species, we require $0 < \alpha < 1$. The dynamics governed by the above equations will conserve the total mass (number of particles) given by $N_1 = \int |\psi_1|^2 \mathrm{d}\mathbf{x}$ and $N_2 = \int |\psi_2|^2 \mathrm{d}\mathbf{x}$. In addition, the total energy (Hamiltonian) of the coupled system

$$H = \int \left[\sum_{i=1}^{2} \left\{ |\nabla\psi_i|^2 + \frac{1}{2}|\psi_i|^4 \right\} + \alpha|\psi_1|^2|\psi_2|^2 \right] \mathrm{d}V \tag{2}$$

will be conserved. Without loss of generality we shall assume that $N_2 \leq N_1$.

Despite the formal reversibility of the above Hamiltonian system, the evolution of the nonlinear waves ψ_1 and ψ_2 is nonintegrable, giving rise to an effective diffusion in phase space. This results in an *irreversible* evolution to thermal equilibrium. By invoking the random phase approximation (assumption of quasi-Gaussian statistics), it is possible to derive closed irreversible kinetic equations that describe the evolution of the system using Weak Turbulence Theory (WTT) [14]. For a homogeneous system, we accomplish this by expressing the order parameters in terms of their Fourier transforms $\psi_1 = \frac{1}{(2\pi)^{3/2}} \int a_{\mathbf{k}}(t) \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}} \mathrm{d}\mathbf{k}$, $\psi_2 = \frac{1}{(2\pi)^{3/2}} \int b_{\mathbf{k}}(t) \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}} \mathrm{d}\mathbf{k}$. Substituting into Eq. (2), we can derive expressions for the spectral number densities $\left\langle a_{\mathbf{k}_1} a^*_{\mathbf{k}_2} \right\rangle = n_1 \delta(\mathbf{k}_1 - \mathbf{k}_2)$; $\left\langle b_{\mathbf{k}_1} b^*_{\mathbf{k}_2} \right\rangle = l_1 \delta(\mathbf{k}_1 - \mathbf{k}_2)$. Provided the nonlinearity in the system is sufficiently weak (i.e. $N_1/V \ll 1$; $N_2/V \ll 1$; $\alpha \ll 1$, where V is the volume of the system), we can derive the kinetic equations

$$\begin{aligned} \partial_t n_k &= \frac{4\pi}{(2\pi)^6} \int ([(n_k + n_1)n_2 n_3 - n_k n_1 (n_2 + n_3)] \\ &\quad + \alpha^2 [(n_k + l_1) l_2 n_3 - n_k l_1 (l_2 + n_3)]) \\ &\quad \times \delta(\mathbf{k} + \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3) \delta(k^2 + k_1^2 - k_2^2 - k_3^2) \mathrm{d}\mathbf{k}_1 \mathrm{d}\mathbf{k}_2 \mathrm{d}\mathbf{k}_3. \\ \partial_t l_k &= \frac{4\pi}{(2\pi)^6} \int ([(l_k + l_1) l_2 l_3 - l_k l_1 (l_2 + l_3)] \\ &\quad + \alpha^2 [(l_k + n_1) n_2 l_3 - l_k n_1 (n_2 + l_3)]) \\ &\quad \times \delta(\mathbf{k} + \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3) \\ &\quad \times \delta(k^2 + k_1^2 - k_2^2 - k_3^2) \mathrm{d}\mathbf{k}_1 \mathrm{d}\mathbf{k}_2 \mathrm{d}\mathbf{k}_3. \end{aligned} \tag{3}$$

These equations conserve $N_1 = (2\pi)^{-3} \int n_{\mathbf{k}}(t) \mathrm{d}\mathbf{k}$, $N_2 = (2\pi)^{-3} \int l_{\mathbf{k}}(t) \mathrm{d}\mathbf{k}$, and the total kinetic energy $E = (2\pi)^{-3} \int k^2 (n_{\mathbf{k}}(t) + l_{\mathbf{k}}(t)) \mathrm{d}\mathbf{k}$. They admit two formal equilibrium solutions; the first corresponding to a uniform distribution $n^{\mathrm{eq}}_{\mathbf{k}} = c_1, l^{\mathrm{eq}}_{\mathbf{k}} = c_2$, and the second given by the Rayleigh–Jeans (RJ) distribution

$$n^{\mathrm{eq}}_{\mathbf{k}} = \frac{T}{k^2 - \mu_1}, \qquad l^{\mathrm{eq}}_{\mathbf{k}} = \frac{T}{k^2 - \mu_2}. \tag{4}$$

Here, T is the thermodynamic temperature, and μ_1 and μ_2 are related to the chemical potentials of the two gases. Eq. (3) satisfies a H-theorem for entropy growth which implies that the RJ distribution will be realised in practice. However, Eq. (4) provides only a formal solution since it leads to non-convergent expressions for N_1, N_2, and the kinetic energy E, as $k \to \infty$. We recall that, for BECs, Eq. (1) is valid in the limit of large occupation numbers where a semi-classical description is valid. When $n_{\mathbf{k}} \sim 1$ and $l_{\mathbf{k}} \sim 1$, Eq. (1) begins to break down and a full quantum mechanical treatment of the problem becomes necessary. To regularise the ultra-violet catastrophe, we introduce a cut-off k_c such that $n^{\mathrm{eq}}(|k_c|) > 1$, $l^{\mathrm{eq}}(|k_c|) > 1$. The physical justification for such a cut-off follows from the fact that, for small occupation numbers, quantum mechanical corrections would modify the RJ distribution to the Bose–Einstein distribution. Since our model is semi-classical, these corrections are not included. A cut-off must, therefore, be imposed by hand. This regularisation of the ultra-violet catastrophe is essential in order for our model to realise an equilibrium state with a finite, non-zero temperature. Such a regularisation has also been used in other studies [4,15]. In practice, this cut-off is introduced by prescribing a particular grid resolution in our simulations. Such a cut-off can only guarantee that the above condition is satisfied at thermodynamic equilibrium, but not necessarily throughout the transients (i.e. throughout the formation and growth of the condensates). Consequently, care must be exercised in interpreting some quantitative aspects of the nonequilibrium state of the system. This problem is related to the well-known bottleneck effect associated with energy spectrum pileup at high wave numbers [22]. Nevertheless, qualitative features concerning the scenario of condensate formation are well described by this approach.

The RJ distribution corresponding to the solution of Eq. (3) is only valid at sufficiently high energies when no condensate is present. At sufficiently low energies, Eq. (3) breaks down very rapidly giving way to the formation of a condensate as elucidated in numerical simulations for a one-component system [4–6] and a two-component system [16]. In the simplest scenario, condensates with zero wavenumbers are formed and are associated with the uniform states provided we are in the phase mixing regime. If the condensates that form are strong in the sense that $(N_1 - n_o)/N_1 \ll 1$, $(N_2 - l_o)/N_2 \ll 1$ ($n_o = |a_o|^2$; $l_o = |b_o|^2$ are the occupation numbers of the condensates in components 1 and 2, respectively), one can describe the nonlinear dynamics at these later times by considering the evolution of small quasiparticle