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Chaotic attractors in incommensurate fractional order systems

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Abstract

In this paper, based on the stability theorems in fractional differential equations, a necessary condition is given to check the existence of 1-scroll, 2-scroll or multi-scroll chaotic attractors in a fractional order system. This condition is proposed for incommensurate order systems in general, but in the special case it converts to the condition given in the previous works for the commensurate fractional order systems. Though the presented condition is only a necessary (and not sufficient) condition for the existence of chaos it can be used as a powerful tool to distinguish for what parameters and orders of a given fractional order system, chaotic attractors can not be observed and for what parameters and orders, the system may generate chaos. It can also be used as a tool to confirm or reject results of a numerical simulation. Some of the numerical results reported in the previous literature are confirmed by this tool.

Keywords: Fractional order system; Incommensurate order systems; Chaos; Chaotic attractor

1. Introduction

Fractional calculus is a mathematical topic with a more than 300 year old history but its application to physics and engineering has been reported only in the recent years. It has been found that in interdisciplinary fields, many systems can be described by fractional differential equations. For instance, fractional derivatives have been widely used in the mathematical modeling of viscoelastic materials [1,2]. Some electromagnetic problems are described using fractional integro-differentiation operators [3]. The anomalous diffusion phenomena in inhomogeneous media can be explained by non-integer derivative based equations of diffusion [4,5]. Another example for an element with fractional order model is the *fractance*. The *fractance* is an electrical circuit with non-integer order impedance [6]. This element has properties lie between resistance and capacitance. Tree fractance [7] and chain fractance [8], as infinite self-similar circuits consisting of resistors and capacitors, are two well known examples of fractances. The resistance–capacitance–inductance (RLC) interconnect model of a transmission line is a fractional order model [9]. Heat conduction as a dynamical process can be more adequately modeled by fractional order models than by their integer order counterparts [10]. In biology, it has been deduced that the membranes of cells of biological organism have fractional order electrical conductance [11] and then are classified in group of non-integer order systems. In economics, it is known that some finance systems can display fractional order dynamics [12]. More examples from fractional order dynamics can be found in [13,14] and references therein.

Study on the fractional order systems has attracted increasing attention in the recent years. For instance, in [15] it has been shown that a limit cycle can be generated in the fractional order Wien bridge oscillator. Dynamics of the fractional order Van der Pol oscillator has been studied in [16]. Existence of a limit cycle for the fractional Brusselator has been shown in [17]. Also, it has been found that some fractional order differential systems can demonstrate chaotic behavior. The fractional order Chua circuit [18], the fractional order Duffing system [19], the fractional order jerk model [20], the fractional order Lorenz system [21], the fractional order Chen system [22], the fractional order Lü system [23], the fractional order Rössler system [24], the fractional order

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Arneodo system [25], the fractional order Newton–Leipnik system [26], the fractional order Genesio–Tesi system [27], the fractional order Ikeda delay system [28] and non-integer order cellular neural networks [29] are well known examples from these kinds of systems. In most of the above mentioned papers existence of chaos has been demonstrated only based on numerical simulation results. Because of low accuracy associated with some of the numerical methods or limitations of them to detect chaos [30,31], wrong results have been reported in special cases. Due to this deficiency, a logical need is observed to develop analytical methods in order to investigate chaos in fractional order systems. One of these methods which is constructed based on the stability analysis in fractional order systems is the subject of this paper. In fact, the work presented here is the extension of our previous work in [30]. In [30], we established a necessary condition through which existence of a 2-scroll attractor in fractional order systems with commensurate order can be checked. In this paper, our aim is to find a similar condition to check the existence of 1-scroll, 2-scroll or multi-scroll chaotic attractors in incommensurate fractional order systems.

This paper is organized as follows. Section 2 summarizes the basic concepts in fractional calculus. In Section 3, two stability theorems on fractional order systems and the related results are presented. Based on these two stability theorems, the required condition by which an incommensurate fractional order system can demonstrate chaos and produce 1-scroll, 2-scroll or multi-scroll attractors is discussed in Section 4. Numerical simulations are presented in Section 5 and finally conclusions in Section 6 close the paper.

2. Basic concepts

Fractional calculus as an extension to ordinary calculus possesses definitions that stem from the definitions existing for ordinary derivatives. Some of the existing definitions for fractional derivatives are described in [14]. The Riemann–Liouville definition is the simplest one and the easiest definition to use. Based on this definition, the αth order fractional derivative of function $f(t)$ with respect to t and the terminal value 0 is given by:

$$\frac{\mathrm{d}^\alpha f(t)}{\mathrm{d}t^\alpha} = \frac{1}{\Gamma(m-\alpha)} \frac{\mathrm{d}^m}{\mathrm{d}t^m} \int_0^t (t-\tau)^{m-\alpha-1} f(\tau)\mathrm{d}\tau, \quad (1)$$

where m is the first integer larger than α, i.e., $m-1 \le \alpha < m$ and $\Gamma(.)$ is the Gamma function,

$$\Gamma(z) = \int_0^\infty t^{z-1} \mathrm{e}^{-t} \mathrm{d}t. \quad (2)$$

The term “terminal value” indicates the lower limit in the integral appeared in (1) and it may be a nonzero value in the general definition of the fractional derivative. The Laplace transform of the Riemann–Liouville derivative is given as follows:

$$L\left\{\frac{\mathrm{d}^\alpha f(t)}{\mathrm{d}t^\alpha}\right\} = s^\alpha L\{f(t)\} - \sum_{k=0}^{m-1} s^k \frac{\mathrm{d}^{\alpha-k-1} f(0)}{\mathrm{d}t^{\alpha-k-1}},$$

$$m-1 < \alpha \le m. \quad (3)$$

Unfortunately, the Riemann–Liouville fractional derivative appears unsuitable to be treated by the Laplace transform technique in that it requires knowledge of the non-integer order derivatives of the function at $t = 0$. The mentioned problem does not exist in the Caputo definition of the fractional derivative. This definition of derivative, which is sometimes called smooth fractional derivative, is described as:

$$\frac{\mathrm{d}^\alpha f(t)}{\mathrm{d}t^\alpha} = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \displaystyle\int_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha+1-m}} \mathrm{d}\tau, \\ \quad m-1 < \alpha < m \\ \dfrac{\mathrm{d}^m}{\mathrm{d}t^m} f(t), \quad \alpha = m, \end{cases} \quad (4)$$

where m is the first integer larger than α. The Laplace transform of the Caputo fractional derivative is:

$$L\left\{\frac{\mathrm{d}^\alpha f(t)}{\mathrm{d}t^\alpha}\right\} = s^\alpha L\{f(t)\} - \sum_{k=0}^{m-1} s^{\alpha-1-k} f^{(k)}(0),$$

$$m-1 < \alpha \le m \in N. \quad (5)$$

Contrary to the Laplace transform of the Riemann–Liouville fractional derivative, only integer order derivatives of function f appear in the Laplace transform of the Caputo fractional derivative. For zero initial conditions, (5) reduces to:

$$L\left\{\frac{\mathrm{d}^\alpha f(t)}{\mathrm{d}t^\alpha}\right\} = s^\alpha L\{f(t)\}. \quad (6)$$

In the rest of this paper, the notation $\mathrm{d}^\alpha/\mathrm{d}t^\alpha$ represents the Caputo fractional derivative of order α.

A linear time invariant fractional order system can be defined by the following state space model:

$$\begin{cases} \dfrac{\mathrm{d}^\alpha x}{\mathrm{d}t^\alpha} = Ax + Bu \\ y = Cx, \end{cases} \quad (7)$$

where $x \in R^n$, $u \in R^m$, $y \in R^p$ are states, inputs, and outputs vectors of the system and $A \in R^{n\times n}$, $B \in R^{n\times m}$, $C \in R^{p\times n}$, and $\alpha = [\alpha_1, \alpha_2, \ldots, \alpha_n]$ indicates the fractional orders, i.e. $\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha} = [\frac{\mathrm{d}^{\alpha_1}}{\mathrm{d}t^{\alpha_1}}, \frac{\mathrm{d}^{\alpha_2}}{\mathrm{d}t^{\alpha_2}}, \ldots, \frac{\mathrm{d}^{\alpha_n}}{\mathrm{d}t^{\alpha_n}}]^{\mathrm{T}}$. If $\alpha_1 = \alpha_2 = \cdots = \alpha_n$, system (7) is called a commensurate order system, otherwise system (7) indicates an incommensurate order system. Now, we state two stability theorems from fractional calculus.

3. Stability theorems

In this section we describe two theorems on fractional order systems and their related results. The first theorem has been given for commensurate fractional order systems.

Theorem 1 (*[32]*). *The following autonomous system:*

$$\frac{\mathrm{d}^\alpha x}{\mathrm{d}t^\alpha} = Ax, \qquad x(0) = x_0, \quad (8)$$

with $0 < \alpha < 1$, $x \in R^n$ *and* $A \in R^{n\times n}$, *is asymptotically stable if and only if* $|\arg(\lambda)| > \alpha\pi/2$ *is satisfied for all*