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Stirring up trouble: Multi-scale mixing measures for steady scalar sources

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Abstract

The mixing efficiency of a flow advecting a passive scalar sustained by steady sources and sinks is naturally defined in terms of the suppression of bulk scalar variance in the presence of stirring, relative to the variance in the absence of stirring. These variances can be weighted at various spatial scales, leading to a family of multi-scale mixing measures and efficiencies. We derive *a priori* estimates on these efficiencies from the advection–diffusion partial differential equation, focusing on a broad class of statistically homogeneous and isotropic incompressible flows. The analysis produces bounds on the mixing efficiencies in terms of the Péclet number, a measure of the strength of the stirring relative to molecular diffusion. We show by example that the estimates are sharp for particular source, sink and flow combinations. In general the high-Péclet-number behavior of the bounds (scaling exponents as well as prefactors) depends on the structure and smoothness properties of, and length scales in, the scalar source and sink distribution. The fundamental model of the stirring of a monochromatic source/sink combination by the random sine flow is investigated in detail via direct numerical simulation and analysis. The large-scale mixing efficiency follows the upper bound scaling (within a logarithm) at high Péclet number but the intermediate and small-scale efficiencies are qualitatively less than optimal. The Péclet number scaling exponents of the efficiencies observed in the simulations are deduced theoretically from the asymptotic solution of an internal layer problem arising in a quasi-static model.

Keywords: Stirring; Mixing; Advection; Diffusion; Eddy diffusion; Turbulent diffusion

1. Introduction

Mixing processes in fluids play a key role in a wide variety of engineering applications and for natural systems such as the ocean and atmosphere. Their theoretical study has been a major focus of research, as indicated by the large number of review articles [1–8]. At the smallest scales mixing is achieved by molecular diffusion processes, but it may be facilitated greatly by stirring. The result of stirring is usually to enhance the effect of molecular diffusion and increase the mixing rate [9–13]. Quantitative understanding of the fundamental features of stirring and its influence on mixing processes is important for the effective modeling, simulation and design or control of these systems.

The "efficiency" of mixing means different things in different contexts. For example the dispersion of an initial distribution by an imposed flow is a transient problem where the temporal approach to the final fully mixed state, rather than the final state itself, is of central interest. Consider for definiteness the homogeneous advection–diffusion equation for a passive scalar field $\theta(\boldsymbol{x}, t)$ stirred by a divergence-free velocity field $\boldsymbol{u}(\boldsymbol{x}, t)$,

$$\frac{\partial \theta}{\partial t} + \boldsymbol{u} \cdot \nabla \theta = \kappa \Delta \theta \tag{1.1}$$

where κ is the molecular diffusivity. If this equation is supplied with initial concentration $\theta(\boldsymbol{x}, 0)$ and applied in an appropriate domain without sources, sinks or scalar flux at the boundaries,

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then the integral of θ is conserved, so without loss of generality it may be taken to vanish from the start. But the L_2-norm $\|\theta(\cdot, t)\|_2$, proportional to the scalar variance in finite volume domains, decreases with time. Indeed, multiplying (1.1) by θ and integrating by parts,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\theta\|_2^2 = -2\kappa\|\nabla\theta\|_2^2, \tag{1.2}$$

indicating an inexorable decay of the variance. Efficient mixing in this transient decay problem means faster decay of the scalar variance. The mixing efficiency of a particular flow could be defined, for example, in terms of its ability to reduce the variance from the initial value to a prescribed value within a specific period of time [14]. Because the right-hand side of (1.2) is proportional to κ, it is evident that molecular processes are ultimately responsible for mixing by this criterion. Even though the stirring field does not appear explicitly in (1.2), the conventional intuition is that material line stretching in the flow can amplify scalar gradients thereby enhancing the molecular mixing rate. Indeed, the velocity's rate-of-strain matrix serves as the local growth rate of the scalar gradient field. These issues are of extreme interest for both theory and applications, but in this paper we are interested in a distinct scenario where different effects are at work.

Mixing a scalar field whose fluctuations are constantly replenished by steady but spatially inhomogeneous sources and sinks is a problem with a long history. Early on, Townsend [15,16] was concerned with the effect of turbulence and molecular diffusion on a line source of temperature, a heated filament. The spatial localization of the source, imposed by experimental constraints, enhanced the role of molecular diffusivity. Saffman [17] also found that molecular diffusion and turbulent diffusion were not simply additive and that higher-order corrections were needed. Durbin [18] and Drummond [19] introduced stochastic particle models to turbulence modeling, and these allowed more detailed studies of the effect of the source on diffusion. Sawford and Hunt [20] pointed out that small sources, such as heated filaments, lead to an explicit dependence of the variance on molecular diffusivity. Many refinements to these models followed, see for instance [21,22] and the review by Sawford [5]. Chertkov et al. [23–27] and Balkovsky and Fouxon [28] treated the case of a random, statistically steady source. Our goal in the present paper is to make the source dependence of the concentration variance more precise by working directly from the advection–diffusion equation, without specifying the underlying turbulent statistics other than basic stationarity and homogeneity assumptions.

When a source of scalar concentration is present, the transient kinetics are of less immediate interest and instead the properties of the (statistical) steady state are of greater relevance. As will be seen, this sustained steady-state dynamics highlights other features of stirring and mixing processes. In comparing the steady-state problem to the transient problem defined by Eq. (1.2), it is important to remember that the long-time asymptotic behavior of the decaying problem is usually irrelevant to the corresponding long-time behavior of the steady-state problem. This is because the continuous replenishing of concentration overwhelms small-amplitude effects observed for long-time decay, such as the 'strange eigenmode' [29–40].

In this paper we consider the stirring and mixing of a passive scalar sustained by a steady source–sink function $s(\boldsymbol{x})$. Given a prescribed divergence-free velocity field $\boldsymbol{u}(\boldsymbol{x}, t)$ and a molecular diffusivity κ, the scalar concentration $\theta(\boldsymbol{x}, t)$ obeys the inhomogeneous advection–diffusion equation

$$\frac{\partial\theta}{\partial t} + \boldsymbol{u}\cdot\nabla\theta = \kappa\Delta\theta + s(\boldsymbol{x}) \tag{1.3}$$

supplemented with initial concentration field $\theta(\boldsymbol{x}, 0)$. We consider a domain without any net scalar flux at the boundaries: the periodic box of size L, i.e., $\boldsymbol{x} \in \mathbb{T}^d$, the d-dimensional torus of volume L^d. The spatial mean of θ is computed immediately,

$$\frac{1}{L^d}\int \theta(\boldsymbol{x}, t)\,\mathrm{d}^d x = \frac{1}{L^d}\int \theta(\boldsymbol{x}, 0)\,\mathrm{d}^d x + t \times \frac{1}{L^d}\int s(\boldsymbol{x})\,\mathrm{d}^d x, \tag{1.4}$$

and deviations from the spatial mean satisfy (1.3) with $s(\boldsymbol{x})$ replaced by $s(\boldsymbol{x}) - L^{-d}\int s\,\mathrm{d}^d x$. So to study the fluctuations we may assume without loss of generality that $\theta(\boldsymbol{x}, 0)$ and $s(\boldsymbol{x})$, and thus also $\theta(\boldsymbol{x}, t)$ have spatial mean zero.

Fluctuations in the scalar concentration are naturally measured in terms of the steady-state variance $\langle\theta^2\rangle$, where we introduce the space–time average. The two averaging operations we use are the time average

$$\overline{F}(\boldsymbol{x}) := \lim_{t\to\infty}\frac{1}{t}\int_0^t F(\boldsymbol{x}, t')\,\mathrm{d}t', \tag{1.5}$$

assuming as necessary that the limit exists, and the space–time average

$$\langle F\rangle := \frac{1}{L^d}\int \overline{F}(\boldsymbol{x})\,\mathrm{d}^d x. \tag{1.6}$$

Effective stirring makes the scalar field more spatially uniform, lowering the variance, and this is the basic mixing effect that we set out to study. Many investigations have been concerned with other statistical properties of the scalar field for this kind of model, such as details of the tails of the probability distribution of θ [2,6]. While these studies present fascinating mathematical and physical issues, in terms of applications they are most likely to be of ultimate use in designing closure approximations, i.e., models of the model, in order to accurately estimate bulk measures of mixing like variance reduction. In this work we focus directly on the suppression of the scalar fluctuations as a primary indicator of mixing.

In terms of the Fourier decomposition of the scalar field,

$$\hat{\theta}_{\boldsymbol{k}}(t) = \frac{1}{L^d}\int \theta(\boldsymbol{x}, t)\,\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\,\mathrm{d}^d x \tag{1.7}$$