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Physica D 235 (2007) 72–81
PHYSICA D
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Method of Riemann surfaces in the study of supercavitating flow around two hydrofoils in a channel

Y.A. Antipov[a,*], V.V. Silvestrov[b]

[a] *Department of Mathematics, Louisiana State University, Baton Rouge, LA 70803, USA*
[b] *Department of Mathematics, Gubkin Russian State University of Oil and Gas, Moscow 119991, Russia*

Available online 2 August 2007

Abstract

In the framework of the Tulin model of supercavitating flow, the problem of reconstructing the free surface of a channel and the shapes of the cavities behind two hydrofoils placed in an ideal fluid is solved in closed form. The conformal map that transforms a parametric plane with three cuts along the real axis into the triple-connected flow domain is found by quadratures. The use of the theory of Riemann surfaces (the Schottky doubles) enables the non-linear model problem to be reduced to two separate Riemann–Hilbert problems on a hyperelliptic surface of genus two. The solution to the first problem is a rational function with certain zeros and poles on a Riemann surface. The second problem is solved in terms of singular integrals with the Weierstrass kernel. The essential singularities of the solution at the infinite points of the surface due to a pole of the kernel are removed by solving a real analogue of the Jacobi inversion problem on the surface. The unknown parameters of the conformal map are recovered from a system of certain algebraic and transcendental equations.

Keywords: Supercavitating flow; Conformal map; Riemann surfaces; Riemann–Hilbert problem

1. Introduction

When an obstacle is placed stationary in a moving fluid the flow breaks away from the barrier along separating streamlines, and a wake (a "dead-zone") forms behind the obstacle. In the case of high-speed flow the wake becomes a vapor-filled cavity. For water under atmospheric pressure for example, this occurs when the speed is 100 f/s or more (lower speeds under reduced pressure) [1]. Cavitating flow has been studied intensively since marine engineers at the end of the 19th century became aware of the serious problem due to cavitation: an increase in pressure causes the cavities to collapse and release energy resulting in a force which may damage submarine propellers when they are operating at certain depths. Cavitation has long been of interest not only in the field of shipbuilding and hydraulic machinery, but also in chemical processing, nuclear physics and medicine (potential bioeffects of ultrasound caused by acoustic cavitation in blood vessels).

Modeling of cavitation is based on the results by Brillouin (1911) (see, e.g. [1]) who proved that the maximum velocity must be attained on the free surface and also that the boundary of a cavity is convex. Good references to the theory of cavitating flow around obstacles are [1–3]. Tulin [4] proposed a model of the cavitating flow which admits the presence of the singularity of the solution at the point say, C, where the two streamlines along the cavity attempt to close it, namely,

$$\log(\mathrm{d}w/\mathrm{d}z) \sim K(w - w_0)^{-1/2}, \quad z \to C, \quad K = \text{const}, \tag{1.1}$$

where $w_0 = w(C)$, $w = w(z)$ is the complex potential of the motion. This condition extends the class of solutions to the governing boundary-value problem which makes possible to reconstruct a flow that meets the condition

$$\oint_L \mathrm{d}z = 0, \tag{1.2}$$

and is therefore single-valued. Here L is the boundary of the cavity combined with the boundary of the hydrofoil.

* Corresponding author. Tel.: +1 225 578 1567; fax: +1 225 5784276.
E-mail address: antipov@math.lsu.edu (Y.A. Antipov).

0167-2789/$ - see front matter
doi:10.1016/j.physd.2007.04.013

For model problems of cavitating flow, numerical and analytical methods are employed. The former technique requires the solution of associated integral equations with the contour to be recovered. The analytical methods for the study of cavitating flow in simply- and double-connected domains are well developed [1,3,5]. These methods are based on the use of a Schwarz–Christoffel transformation to map a circle, a quadrant, or a plane with a cut (simply-connected flow), or an annulus, a rectangular, or a plane with two cuts on the real axis (double-connected flow) into the flow domain. A closed-form solution to the model problem can then be found in terms of elementary and elliptic functions for a simply- and double-connected flow, respectively.

In the case of free boundary problems in a triple-connected flow domain, the problem of fluid mechanics can be formulated [6] as the Hilbert boundary-value problem with a piecewise constant coefficient on three cuts along the real axis. The actual problem solved in [6] concerned non-cavitating flow of a fluid around a single foil in a half-plane with a free surface. A method of Riemann surfaces for an elasticity problem [7] on a system of cracks along an interface with mixed boundary conditions was extended for the problem of cavitating flow in the whole plane around three foils [8].

In this paper we model the steady flow of an ideal fluid in a channel with a free surface when two plates are held stationary. The main mathematical tool used is the method of the Riemann–Hilbert problem on a Riemann surface. This technique was applied and developed [9–11] for the solution of model problems of acoustic and electromagnetic scattering from a perforated sandwich panel and an anisotropic impedance half-plane. In Section 2 we formulate the problem using the non-linear model of cavitating flow by Tulin. Section 3 maps the problem into two Hilbert problems on three cuts on the real axis. One of the cuts is the image of the boundary of the channel (a portion of it is a free boundary and therefore is unknown). The other two cuts are the images of the cavities whose boundaries formed by the hydrofoils and the unknown convex contours. The derivative of the conformal map $z = f(\zeta)$ is represented as a quotient of two functions, $\mathrm{d}w/\mathrm{d}z$ and $\mathrm{d}w/\mathrm{d}\zeta$, where w is a complex potential of the flow. In Section 4 the function $\mathrm{d}w/\mathrm{d}\zeta$ is found as a rational function with certain zeros and poles on a hyperelliptic surface of genus two. To define the function $\omega(\zeta) = \log(V_\infty^{-1}\mathrm{d}w(z)/\mathrm{d}z)$ (V_∞ is the speed at infinity), in Section 5 we reduce the Hilbert problem on the three cuts with a piecewise constant coefficient to the Riemann–Hilbert problem on the Riemann surface introduced in Section 4. Its solution is found by quadratures in terms of singular integrals with the Weierstrass kernel. Initially, it has an inadmissible exponential growth at infinity. The conditions which make the solution bounded at infinity are written as the Jacobi inversion problem for hyperelliptic integrals. This non-linear problem requires finding two points on the surface and four integers. The solution of the Jacobi problem is found in closed form by reducing it to a system of two algebraic equations with the right-hand side expressed through the Riemann θ-function of the surface. Section 6 writes down additional conditions to be satisfied in order to fix 21 unknown real parameters. The system of equations for the unknowns consists of 8 linear and 13 non-linear relations which are algebraic and transcendental equations. Finally, equations of the free surface of the channel and the boundaries of the two cavities are found by quadratures.

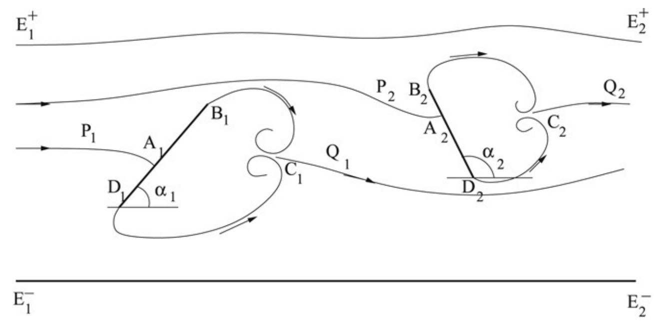

Fig. 1. Flow domain $\mathcal{D}$.

2. Formulation

Let two hydrofoils B_1D_1 and B_2D_2 (Fig. 1) be placed in an incompressible gravity-free fluid which is moving steadily and irrotationally in a channel. The bottom $\{-\infty < x < \infty, y = 0\}$ of the channel is solid, and its upper boundary is a free surface. Far away from the hydrofoils, the flow is uniform with velocity $\mathbf{v} = (V_\infty, 0)$ across the channel of depth h. It is assumed that at the ends B_j and D_j ($j = 1, 2$) the jets break away from the hydrofoils, and cavities $B_jC_jD_j$ ($j = 1, 2$) form behind the foils. The cavities are convex and bounded but not closed. The unknown boundaries of the cavities are streamlines. In the framework of the model considered, the velocity vector is constant and prescribed on the boundaries of the cavities (the constants are not necessarily the same). The loops $A_jB_jC_jD_jA_j$ are smooth in a neighborhood of the points B_j and D_j. At the stagnation points A_j (unknown *a priori*), the flow branches and the velocity vector vanishes.

Under these assumptions, the model problem of fluid mechanics is reduced to that of finding a complex potential of the motion $w(z) = \phi + \mathrm{i}\psi$ in the 3-connected domain say, $\mathcal{D}$, occupied by the fluid (the physical domain), together with boundary conditions of the form

$$\operatorname{Im} w(z) = \begin{cases} W_0^\pm, & z \in E_1^\pm E_2^\pm \\ W_j, & z \in L_j, \end{cases} \qquad j = 1, 2,$$

$$\left|\frac{\mathrm{d}w}{\mathrm{d}z}\right| = \begin{cases} V_\infty, & z \in E_1^+ E_2^+ \\ V_j, & z \in B_jC_jD_j, \end{cases} \qquad j = 1, 2,$$

$$\arg\frac{\mathrm{d}w}{\mathrm{d}z} = \begin{cases} 0, & z \in E_1^- E_2^- \\ -\alpha_j, & z \in A_jB_j \\ \pi - \alpha_j, & z \in A_jD_j, \end{cases} \qquad j = 1, 2. \tag{2.1}$$

Here $W_0^\pm$ and W_j ($j = 1, 2$) are some constants, $\mathrm{d}w/\mathrm{d}z = v_x - \mathrm{i}v_y$, v_x and v_y are the velocity components, V_j are positive constants defined by the Bernoulli equation

$$\frac{1}{2}(V_j^2 - V_\infty^2) + \frac{p_j - p_\infty}{\rho} = 0, \quad j = 1, 2, \tag{2.2}$$

p_∞ and p_j are the pressure at infinity and in the cavities,