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Physica D 221 (2006) 72–83
PHYSICA D
www.elsevier.com/locate/physd

Phase diagram of the random frequency oscillator: The case of Ornstein–Uhlenbeck noise

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Received 7 October 2005; received in revised form 7 June 2006; accepted 14 July 2006
Available online 10 August 2006
Communicated by A. Pikovsky

Abstract

We study the stability of a stochastic oscillator whose frequency is a random process with finite time memory represented by an Ornstein–Uhlenbeck noise. This system undergoes a noise-induced bifurcation when the amplitude of the noise grows. The critical curve, that separates the absorbing phase from an extended non-equilibrium steady state, corresponds to the vanishing of the Lyapunov exponent that measures the asymptotic logarithmic growth rate of the energy. We derive various expressions for this Lyapunov exponent by using different approximation schemes. This allows us to study quantitatively the phase diagram of the random parametric oscillator.

Keywords: Random processes; Fokker–Planck equations; Colored noise; Stochastic analysis methods

1. Introduction

Noise can modify drastically the phase diagram of a dynamical system [1–3]. Because of stochastic fluctuations of the control parameter, the critical value of the threshold can change and noise can delay or favor a phase transition [4]. In the first case, randomness can be useful as a stabilizing mean and in the second, noise may help trigger a phase transition that is otherwise very difficult to achieve; for example, in the dynamo effect, the role of the noise generated by fluid turbulence is not well understood at present and it is possible that the critical magnetic Reynolds number decreases with noise [5]. In certain cases, a physical system subject to noise undergoes bifurcations into states that have no deterministic counterparts: the stochastic phases generated by randomness have specific characteristics (such as scaling behavior or critical exponents) that define new universality classes [6,7].

One of the simplest systems that can be used as a paradigm for the study of noise-induced phase transitions is the random frequency oscillator [8,9]. For instance, in practical engineering problems, the Duffing oscillator with random frequency has been used as a model to study stability of structures subject to random external forces, such as earthquakes, wind or ocean waves [10–13]. Whereas a deterministic oscillator with damping evolves towards the unique equilibrium state of minimal energy, the behavior changes if the frequency of the oscillator is a time-dependent variable. Due to continuous energy injection into the system through the frequency variations, the system may sustain non-zero oscillations even in the long time limit. The case when the frequency is a periodic function of time is the classical problem of parametric resonance known as the Mathieu oscillator; the phase diagram is obtained by calculating the Floquet exponents defined as the characteristic growth rates of the amplitude of the system [14]. When the frequency of the pendulum is a random process, the role of the Floquet exponents is taken over by the Lyapunov exponents [15,16]. The system undergoes a bifurcation when the largest Lyapunov exponent, defined as the growth rate of the logarithm of the energy, changes its sign. Thus, the Lyapunov exponent vanishes on the critical surface that separates the phases in the parameter space. This criterion involving the sign of the Lyapunov exponent has a firm mathematical basis and clarifies the ambiguities that were found in the study of the stability of higher moments [15,17].

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0167-2789/$ - see front matter © 2006 Published by Elsevier B.V.
doi:10.1016/j.physd.2006.07.013

In a recent work [18], we have carried out an analytic study of the phase diagram of the random oscillator driven by a Gaussian white noise frequency. We have shown [19] that, in the case of an inverted pendulum, the unstable fixed point can be stabilized by noise and a noise-induced reentrant transition occurs. These results are based on an exact formula for the Lyapunov exponent [20–22]. In the present work, we intend to study the phase diagram of an oscillator whose frequency is a random process with finite time memory. More precisely, we consider here the case of an Ornstein–Uhlenbeck noise of correlation time τ. From a physical point of view, the influence of a finite correlation time on the phase diagram is an interesting open question: does a finite correlation time favor or hinder a noise-induced transition? In particular, we wish to determine how the shape of the transition curve is modified when the noise is colored. In the white noise case, the asymptotic behavior of the critical curve when the amplitude of noise is either very small or very large is known explicitly and presents a simple scaling behavior [18]. How do these scalings change when the noise is correlated in time?

Due to the finite correlation time of the noise, the random oscillator is a non-Markovian random process and there exists no closed Fokker–Planck equation that describes the dynamics of the Probability Distribution Function (PDF) in the phase space. This non-Markovian feature hinders an exact solution in contrast with the white noise case where a closed formula for the Lyapunov exponent was found. We shall therefore have to rely on various approximations to carry out an analytical study of the phase diagram. The results obtained by different approximations will be compared with numerical results and with an exact small noise perturbative expansion. The various approximations have different regions of validity in the parameter space: this allows us to derive a fairly complete picture of the phase diagram of the random oscillator subject to an Ornstein–Uhlenbeck multiplicative noise.

The outline of this work is as follows. In Section 2, we derive general results about the stochastic oscillator with random frequency: thanks to dimensional analysis, we reduce the dimension of the parameter space from four to two and show how the Lyapunov exponent can be calculated by using an effective first order Langevin equation; we also recall the exact results for white noise. In Section 3, we rederive the rigorous functional evolution equation of PDF; although this equation is purely formal and is not closed (it involves a hierarchy of correlation functions), it will be used as a systematic basis for various approximations; we also carry out an exact perturbative expansion of the Lyapunov exponent in the small noise limit. In Section 4, we consider a mean-field type approximation known as the ‘decoupling Ansatz’ which provides a simple expression for the colored noise Lyapunov exponent in terms of the white noise Lyapunov exponent. In Section 5, we consider two small correlation time approximations that both lead to an effective Markovian evolution: we show that these approximations are fairly accurate in the small noise regime. In Section 6, we investigate the large correlation time limit by performing an adiabatic elimination: this approximation is quite suitable for the large noise regime. The last section is devoted to a synthesis and a discussion of our results.

2. General results

2.1. The random harmonic oscillator and the Lyapunov exponent

A harmonic oscillator with a randomly varying frequency can be described by the following equation

$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + (\omega^2 + \xi_0(t))x = 0, \tag{1}$$

where $x(t)$ is the position of the oscillator at time t, γ the (positive) friction coefficient and ω the mean value of the frequency. We assume that the frequency fluctuations $\xi(t)$ are modelized by an Ornstein–Uhlenbeck process of amplitude $\mathcal{D}_0$ and of correlation time τ_0. The long time behavior of $x(t)$ is characterized by the Lyapunov exponent defined as

$$\begin{aligned}\Lambda(\omega, \gamma, \mathcal{D}_0, \tau_0) &= \lim_{t\to\infty}\frac{1}{2t}\left\langle \log\left(\frac{\dot{x}^2}{2} + \frac{x^2}{2}\right)\right\rangle \\ &= \lim_{t\to\infty}\frac{1}{2t}\langle \log E\rangle, \end{aligned}\tag{2}$$

where the brackets $\langle\rangle$ indicate an averaging over realizations of the noise between 0 and t, i.e., an averaging with respect to the Probability Distribution Function (PDF) $P_t(x, \dot{x})$; the quantity E is the energy of the system.

Taking the time unit to be ω^{-1}, we obtain the following dimensionless parameters,

$$\alpha = \frac{\gamma}{\omega}, \quad \mathcal{D}_1 = \frac{\mathcal{D}_0}{\omega^3}, \quad \tau_1 = \omega\tau_0. \tag{3}$$

In terms of these parameters, Eq. (1) becomes

$$\frac{d^2x}{dt^2} + \alpha\frac{dx}{dt} + (1 + \xi(t))x = 0. \tag{4}$$

The Ornstein–Uhlenbeck noise $\xi(t)$ now has an amplitude $\mathcal{D}_1$ and a correlation time τ_1 and can be generated from the following linear stochastic differential equation:

$$\frac{d\xi(t)}{dt} = -\frac{1}{\tau_1}\xi(t) + \frac{1}{\tau_1}\eta(t), \tag{5}$$

$\eta(t)$ being a Gaussian white noise of zero mean value and of amplitude $\mathcal{D}_1$. In the stationary limit, $\xi(t)$ has exponentially decaying time correlations:

$$\langle\xi(t)\xi(t')\rangle = \frac{\mathcal{D}_1}{2\tau_1}\exp(-|t - t'|/\tau_1). \tag{6}$$

When $\tau_1 \to 0$, the process $\xi(t)$ becomes identical to the white noise. In terms of the dimensionless parameters, the Lyapunov exponent is given by

$$\Lambda(\omega, \gamma, \mathcal{D}_0, \tau_0) = \omega\Lambda(\alpha, \mathcal{D}_1, \tau_1). \tag{7}$$