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Halo orbits around the collinear points of the restricted three-body problem

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HIGHLIGHTS

- We consider halo orbits around the collinear Lagrangian–Eulerian points.
- An analytical estimate of the bifurcation threshold to halo orbits is obtained.
- The method is based on a normal form adapted to the synchronous resonance.
- A reduction to the central manifold is then performed.
- We make a comparison with available numerical data.

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ABSTRACT

We perform an analytical study of the bifurcation of the halo orbits around the collinear points L_1, L_2, L_3 for the circular, spatial, restricted three-body problem. Following a standard procedure, we reduce to the center manifold constructing a normal form adapted to the synchronous resonance. Introducing a detuning, which measures the displacement from the resonance and expanding the energy in series of the detuning, we are able to evaluate the energy level at which the bifurcation takes place for arbitrary values of the mass ratio. In most cases, the analytical results thus obtained are in very good agreement with the numerical expectations, providing the bifurcation threshold with good accuracy. Care must be taken when dealing with L_3 for small values of the mass-ratio between the primaries; in that case, the model of the system is a singular perturbation problem and the normal form method is not particularly suited to evaluate the bifurcation threshold.

1. Introduction

In the *circular, spatial, restricted three-body problem* (hereafter CSR3BP), in the synodic frame, the *collinear* equilibrium points discovered by Euler are located on the line joining the two primaries: L_1 lies within them, L_2, L_3 are outside the interval joining the primaries. For values of the mass ratio μ of the primaries corresponding to usual applications like the barycenter–Sun system (namely, the system describing the interaction between the Earth–Moon barycenter and the Sun) or the Earth–Moon case, the equilibria L_1, L_2 are close to the smaller primary, while L_3 is quite far on the opposite side of the larger primary. Moreover, two cases are of special interest: $\mu = 0$ and $\mu = 1/2$. The limit $\mu \to 0$ can be interpreted as a *Hill's problem* for L_1 and L_2, where the two equilibria tend to an equal distance from the smaller primary and the problem is equivalent to let one of the primaries go to infinity as in the classical lunar theory developed by G.W. Hill in [1]; in the case of L_3, again in the limit $\mu \to 0$, we will speak about a *quasi-Kepler problem*, since it is equivalent to a nearly two-body problem in the rotating frame. On the opposite side of the mass parameter range, namely the case of equal masses, i.e. $\mu = 1/2$, we find that the equilibrium L_1 is midway from the primaries and L_2, L_3 are at the same distance from the primaries on each side. The case of such large mass ratio is typically applicable to binary stars or some exotic exo-planetary systems with very large planets.

Overall, in the whole range $\mu \in (0, 1/2]$, we get a very rich dynamical setting with several peculiar phenomena (stability–instability transitions, bifurcations, etc.) characterizing the non-integrable Hamiltonian system associated to the CSR3BP. As it

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is well known, the collinear points are linearly unstable. However, since the seminal paper by C. Conley [2] on the so-called transit-orbits through L_1, much attention has been devoted to the use of the collinear points for space missions [3], thanks to the fact that the unstable behavior can be easily controlled for a reasonable time-span. Moreover, the characteristics of the evolving unstable/stable pathways offer a structure that can be suitably exploited to design transfer orbits between different regions of the phase space.

On the same ground, the unstable dynamics offers a *clean* environment free of debris and dust. Indeed, a neighborhood of L_1 can be considered a privileged position to observe the Sun, while L_2 is very good to observe the Universe shielding the Sun through the Earth.

To study the dynamics around these points, numerical methods provide high accuracy and fast algorithms to follow the evolution from given initial conditions. However, the analytical theory gives a deeper insight into the nature of the global behavior in a neighborhood of these solutions, so to get a comprehensive description of the dynamics in the whole mass range [4–6]. For example, the phenomena connected with low-order resonances around equilibria play a leading role in shaping the phase-space structure and provide a coarse picture of the global dynamics. Finer details like secondary resonances or heteroclinic intersections are often confined to small portions of the phase-space and usually require dedicated numerical experiments.

The aim of this paper is to present an analytical method to predict the bifurcation thresholds of the *halo orbits* around the three collinear points for arbitrary values of the mass ratio: in the present setting this is the most prominent effect of the resonance in the whole interval between the two extreme cases $\mu = 0$ and $\mu = 1/2$. According to Lyapunov's center theorem, each collinear point generates a pair of one-parameter families of periodic orbits (the nonlinear *normal modes*), to which we refer as the planar and the vertical Lyapunov families. By halo orbit we intend the family of periodic orbits, which arise at the first 1:1 bifurcation from the planar Lyapunov family. A resonant perturbation theory allows us to investigate the halo family and to determine the value of the energy at which the bifurcation from the planar Lyapunov family, namely the *horizontal* normal mode, takes place [7]. We also construct a normal form to perform the center manifold reduction (see [4]), which yields an integrable approximation of the dynamics (compare with [8]). The unperturbed linear dynamics on the 2-dimensional center manifold is characterized by almost equal values of the frequencies for all mass ratios. Therefore it is natural to introduce a *detuning* parameter, which describes the departure from the exact resonance [9,10]. By increasing the energy, the bifurcations of the 1:1 resonant periodic orbits from the normal modes can be expressed as series expansions in the detuning.

Our results show that for L_1 and L_2 the prediction of the energy threshold of the bifurcation is very accurate (up to the fourth decimal digit), when compared with numerical data available in the literature (see [11,12]), even limiting just to a second-order computation; however, we make the effort of computing higher orders to look for the best agreement (if any) with available data. Moreover, our strategy allows us to improve previous analytical approaches based on Lindstedt series [13] and to determine first order approximations of the initial conditions for the first bifurcating orbits. In the case of L_3, the peculiar nature of the dynamics around it [14], especially when the mass ratio is less than the Earth–Moon value, gives much less accurate results, but our approach is still useful for a qualitative understanding. In this respect, we provide an explanation for the results concerning L_3 in terms of the optimal order of the Birkhoff normalization procedure applied to a singular perturbation problem.

This work is organized as follows. In Section 2 we present the equations of motion and the location of the collinear points of the CSR3BP. The corresponding Hamiltonian is diagonalized, normalized and reduced to the center manifold in Section 3. In Section 4 we provide analytical formulae for the bifurcation thresholds at different orders of normalization. In Section 5 we present the results of our analytical approach and we compare them with the corresponding numerical values. Section 6 provides some conclusions on the results of the present work.

2. Collinear points in the three-body problem

We consider a synodic reference frame centered in the barycenter of the primaries, which are denoted as $\mathscr{P}_1$, $\mathscr{P}_2$, and rotating with the angular velocity of the primaries. The X axis is set along the line joining $\mathscr{P}_1$ and $\mathscr{P}_2$, the Z axis along the angular momentum and the Y axis in such a way to have a positively oriented frame. We normalize the units of measure so that the gravitational constant as well as the sum of the masses of the primaries are unity. Let us rename μ the mass of the smaller primary; then, with the previous normalization it results that the larger primary is located[1] at $(\mu, 0, 0)$, while the smaller one is at $(\mu - 1, 0, 0)$. The equations of motion of a third small body in the synodic reference frame admit five equilibrium points discovered by L. Euler and J.-L. Lagrange: the triangular and the collinear points (see, e.g., [15,16]). The triangular points L_4 and L_5 are linearly stable whenever μ is smaller than a threshold, called *Routh's value*. On the contrary, the collinear points L_1, L_2, L_3 are shown to be always linearly unstable.

Let us define the kinetic moments P_X, P_Y, P_Z as

$$P_X = \dot{X} - Y, \qquad P_Y = \dot{Y} + X, \qquad P_Z = \dot{Z};$$

the initial Hamiltonian function describing the motion of the third body is given by

$$H^{(IN)}(P_X, P_Y, P_Z, X, Y, Z) = \frac{1}{2}(P_X^2 + P_Y^2 + P_Z^2) + YP_X - XP_Y - \frac{1-\mu}{r_1} - \frac{\mu}{r_2}, \tag{1}$$

where r_1, r_2 denote the distances from the primaries:

$$r_1 = \sqrt{(X-\mu)^2 + Y^2 + Z^2}, \qquad r_2 = \sqrt{(X-\mu+1)^2 + Y^2 + Z^2}.$$

Let us introduce the scalar function, sometimes called *pseudo-potential* (compare with [16]):

$$\Omega(X, Y, Z) \equiv \frac{1}{2}(X^2 + Y^2) + \frac{1-\mu}{r_1} + \frac{\mu}{r_2};$$

then, the equations of motion can be written in compact form as

$$\ddot{X} - 2\dot{Y} = \frac{\partial \Omega}{\partial X},$$
$$\ddot{Y} + 2\dot{X} = \frac{\partial \Omega}{\partial Y},$$
$$\ddot{Z} = \frac{\partial \Omega}{\partial Z}.$$

Next we translate the origin so that it coincides with a collinear point; to this end, we determine the distance γ_j, $j = 1, 2, 3$, of the collinear equilibria from the closest primary as the solution of the fifth order Euler's equations (see, e.g., [4]):

$$\gamma_1^5 - (3-\mu)\gamma_1^4 + (3-2\mu)\gamma_1^3 - \mu\gamma_1^2 + 2\mu\gamma_1 - \mu = 0 \quad \text{for } L_1,$$
$$\gamma_2^5 + (3-\mu)\gamma_2^4 + (3-2\mu)\gamma_2^3 - \mu\gamma_2^2 - 2\mu\gamma_2 - \mu = 0 \quad \text{for } L_2,$$
$$\gamma_3^5 + (2+\mu)\gamma_3^4 + (1+2\mu)\gamma_3^3 - (1-\mu)\gamma_3^2 - 2(1-\mu)\gamma_3 - (1-\mu) = 0 \quad \text{for } L_3.$$

[1] Notice that with the present convention the equilibrium point L_2 is located to the left of the smaller primary, L_1 lies between the primaries and L_3 stands at the right of the larger primary.