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Orbital stability of periodic traveling-wave solutions for the regularized Schamel equation

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HIGHLIGHTS

- We study the regularized Schamel equation.
- Existence of a one-parameter family of periodic traveling-wave solutions is proved.
- Orbital stability in the energy space is provided.
- Global well-posedness for the Cauchy problem in the energy space is established.

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ABSTRACT

In this work we study the orbital stability of periodic traveling-wave solutions for dispersive models. The study of traveling waves started in the mid-18th century when John S. Russel established that the flow of water waves in a shallow channel has constant evolution. In recent years, the general strategy to obtain orbital stability consists in proving that the traveling wave in question minimizes a conserved functional restricted to a certain manifold. Although our method can be applied to other models, we deal with the regularized Schamel equation, which contains a fractional nonlinear term. We obtain a smooth curve of periodic traveling-wave solutions depending on the Jacobian elliptic functions and prove that such solutions are orbitally stable in the energy space. In our context, instead of minimizing the augmented Hamiltonian in the natural codimension two manifold, we minimize it in a "new" manifold, which is suitable to our purposes.

1. Introduction

This paper sheds new contributions in the sense of obtaining orbital stability of periodic traveling waves for nonlinear dispersive models. Although, we pay particular attention to the regularized Schamel equation,

$$u_t - u_{xxt} + (u + |u|^{3/2})_x = 0, \tag{1.1}$$

such ideas can be applied to a large class of dispersive models (see [1,2]).

The Korteweg–de Vries (KdV) equation,

$$u_t + u_{xxx} + \frac{1}{2}(u^2)_x = 0, \tag{1.2}$$

the regularized long-wave or Benjamin–Bona–Mahony (BBM) equation

$$u_t - u_{xxt} + \left(u + \frac{1}{2}u^2\right)_x = 0, \tag{1.3}$$

and their various modifications are widely used models describing the propagation of nonlinear waves. Originally, (1.3) was derived by Benjamin, Bona, and Mahony in [3] as an alternative model to the KdV equation for small-amplitude, long wavelength surface water waves.

For one hand, (1.1) can be viewed as a regularized version of the Schamel equation

$$u_t + u_{xxx} + (u + |u|^{3/2})_x = 0, \tag{1.4}$$

in much the same way that the BBM equation can be regarded as a regularized version of the KdV equation. Eq. (1.4) was derived by Schamel [4,5] as a model to describe the propagation of weakly

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nonlinear ion acoustic waves which are modified by the presence of trapped electrons.

Both KdV and BBM equations also model the one dimensional waves in a cold plasma (see e.g., [6]), with the difference that BBM equation describes much better the behavior of very short waves (see also [7]). The approach in [6] is based on the use of approximate Hamiltonians. So, depending on the region of validity, a new equation for nonlinear ion waves is obtained. In a general setting, the canonical equations for one-dimensional waves, in dimensionless form, read as

$$\begin{cases} u_t - u_{xxt} = -\partial_x \dfrac{\delta H_a}{\delta v}, \\ v_t - v_{xxt} = -\partial_x \dfrac{\delta H_a}{\delta u}, \end{cases} \tag{1.5}$$

where $H_a = H_a(u, v)$ is the approximate Hamiltonian and u and v are related to the ion mass density and ion mass velocity, respectively. If we take the Hamiltonian to be

$$H_a = \frac{1}{2}\int \left(u^2 + v^2 + \frac{4}{5}\mathrm{sgn}(u)|u|^{5/2}\right) dx$$

we see that (1.5) reduces to

$$\begin{cases} u_t - u_{xxt} = -\partial_x(v), \\ v_t - v_{xxt} = -\partial_x(u + |u|^{3/2}). \end{cases} \tag{1.6}$$

By putting $u+v = w$, $u-v = z$, adding the equations in (1.6), and neglecting z, we find (up to constants) the BBM-like approximation for unidirectional waves as in (1.1) (in the variable w).

Remark 1.1. Although we will not discuss here, the region of validity of H_a can be addressed as in [8]. It is to be observed that the approximate Hamiltonian may be corrected to

$$H_a = \frac{1}{2}\int \left(u^2 + v^2 + uv^2 + \frac{4}{5}\mathrm{sgn}(u)|u|^{5/2}\right) dx.$$

By following the above steps, we obtain the equation

$$w_t - w_{xxt} = -\partial_x(w + aw^2 + b|w|^{3/2}), \tag{1.7}$$

where a and b are real constants. The nonlinearity appearing in (1.7) is in agreement with [5] (see also [9] and references therein for more recent results in this direction), where the author observed that in some physical situations, the nonlinearity in (1.4) should be corrected to $(u^2 + |u|^{3/2})_x$. Following the ideas of our work, one can also study the existence and orbital stability of periodic traveling waves to (1.7).

Eqs. (1.1) and (1.4) as less studied than (1.3) and (1.2), mainly because the fractional power in the nonlinear part brings a lot of difficulties which, in several aspects, cannot be handled with standard techniques. We point out, however, that the spectral stability of periodic traveling-wave solutions for (1.4) was studied in [10].

From the mathematical point of view, the generalized BBM equation

$$u_t - u_{xxt} + (f(u))_x = 0 \tag{1.8}$$

has become a major topic of study in recent years and much effort has been expended on various aspects of (1.8). The issues include the initial-value (initial–boundary-value) problem, existence and stability of solitary and periodic traveling waves and global behavior of solutions. Thus, (1.1) can also be viewed as (1.8) with $f(u) = u + |u|^{3/2}$. In this context, (1.1) has appeared, for instance, in [11] where the authors study the initial-value problem in the usual L^2-based Sobolev spaces and the nonlinear stability of solitary traveling waves (see also [12–14] and references therein).

Our main goal in this work is to establish the existence and orbital stability of an explicit family of periodic traveling-wave solutions associated with (1.1). The traveling waves we are interested in are of the form $u(x,t) = \phi(x - ct)$, where ϕ is a periodic function of its argument and $c > 1$ is a real parameter representing the wave speed. By replacing this form of u in (1.1), one sees that ϕ must solve the nonlinear ODE

$$-c\phi'' + (c-1)\phi - \phi^{3/2} + A = 0, \tag{1.9}$$

where A is an integration constant.

The constant A in (1.9) plays a crucial role in the theory of nonlinear stability. Indeed, assume we are dealing with an invariant by translation Hamiltonian nonlinear evolution equation of the form $u_t = J\nabla E(u)$, where E is the energy functional. Suppose the associated periodic traveling waves satisfy a conservative equation like

$$-\phi'' + h(\phi, c, A) = 0, \tag{1.10}$$

for some smooth function h. Here c represents the wave speed and A is an arbitrary constant. On one hand, when $A = 0$ the, by now, classical stability theories [15–18] pass to showing that ϕ is a local minimum of E restrict to a suitable manifold depending on the conserved quantity originated by translation invariance, say, Q. At this point, the coercivity of the functional $\widetilde{\mathcal{F}} = E + cQ$ develops a fundamental role and many works concerning the stability of periodic waves have appeared in the literature (see e.g., [1,19–30] and references therein). On the other hand, the situation when $A \neq 0$ is a little bit more delicate; the minimization of $\widetilde{\mathcal{F}}$ is not enough to produce the desired results and, in general, we need to add an extra conserved quantity to $\widetilde{\mathcal{F}}$. Thus, it is reasonable to work with a functional having the form $\mathcal{F} = E + cQ + AV$, where V is another conservation law. The quantity V in general does not come from an invariance of the equation. As a consequence, the theories mentioned above cannot be directly applied and this forces us to revisit its core in order to cover the nonlinear stability in such situations (see also [21,25]).

Although in a different way, the case $A \neq 0$ was addressed, for instance in [13,31,32], where the orbital stability of a three- or two-parameter family of periodic traveling waves associated with KdV-type and generalized BBM equations were established (see also [25]). In these works, the authors do not use the explicit form of the waves and prove the existence of local families of traveling waves by using the standard theory of ODE's. Their method has the advantage that it avoids a lot of hard calculations which appear when explicit solutions are studied. However, as we will see below, our approach has the advantage that we can prove the needed spectral properties in a very simple way.

Let us now turn attention to the main steps of our constructions. As we already mentioned, our aim consists in making some changes in the classical theories developed in [15–17], in order to deal with explicit periodic solutions of (1.9) obtained when $A \neq 0$. With this in mind, we prove that (1.9) has a solution of the form

$$\phi(\xi) = (\alpha + \beta \mathrm{CN}^2(\gamma\xi, k))^2,$$

where CN denotes the Cnoidal Jacobi elliptic function, $k \in (0,1)$ represents the elliptic modulus and α, β, and γ are suitable constants depending, a priori, on c and A. After some algebraic manipulations, one can write all parameters in terms of k, so that a smooth curve of explicit L-periodic solutions $k \in J \to \phi_k$ can be obtained. In particular, the parameters c and A can also be written as functions of k.

The conserved quantities appearing here are

$$\begin{aligned} E(u) &= \frac{1}{2}\int_0^L \left(u_x^2 - \frac{4}{5}\mathrm{sgn}(u)|u|^{\frac{5}{2}}\right) dx, \\ Q(u) &= \frac{1}{2}\int_0^L (u^2 + u_x^2) dx, \end{aligned} \tag{1.11}$$