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A geometric singular perturbation approach for planar stationary shock waves

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ABSTRACT

The non-linear non-equilibrium nature of shock waves in gas dynamics is investigated for the planar case. Along each streamline, the Euler equations with non-equilibrium pressure are reduced to a set of ordinary differential equations defining a slow–fast system, and geometric singular perturbation theory is applied. The proposed theory shows that an orbit on the slow manifold corresponds to the smooth part of the solution to the Euler equation, where non-equilibrium effects are negligible; and a relaxation motion from the unsteady to the steady branch of the slow manifold corresponds to a shock wave, where the flow relaxes from non-equilibrium to equilibrium. Recognizing the shock wave as a fast motion is found to be especially useful for shock wave detection when post-processing experimental measured or numerical calculated flow fields. Various existing shock detection methods can be derived from the proposed theory in a rigorous mathematical manner. The proposed theory provides a new shock detection method based on its non-linear non-equilibrium nature, and may also serve as the theoretical foundation for many popular shock wave detection techniques.

1. Introduction

Mathematically, a shock wave is defined as the discontinuity of the solution of hyperbolic conservation [1, p. 15]. In gas dynamics, the specific form of the conservation law is the Euler equation, which is an inviscid simplification of the dissipative Navier–Stokes' equation with Fourier's law, or the NSF model. The physical structure of a shock wave is complicated due to the thermodynamically non-equilibrium processes within it [2–5]. Even the NSF model is invalid in physical shock waves because the continuum assumption does not hold for non-equilibrium process. However, the Euler system can successfully describe this phenomenon, as described by Majda [6, p. 8]. The jump (Rankine–Hugoniot) condition coupled with an appropriate entropy condition is introduced to the Euler system to "incorporate the effects of the small scale diffusion on the large scale quantities without resolving the small scale effects in detail". The complicated non-equilibrium process can be well approximated by the jump condition, which is a set of equations. If diffusion effects are included, a shock wave becomes a thin region with extremely large gradients of flow variables, which leads to a finer description of the shock wave, and carefully modeled diffusion effects could produce an accurate resolution of shock wave structure, although the NSF model is physically invalid in the shock region [3]. The corresponding mathematical terminology is shock layer or viscous shock profile, which is discussed in [1, pp. 265–269] and the references therein, including the classical papers by Weyl [7], Gilbarg [8] and the multidimensional discussions by Guès et al. [9–11].

The viscous model can provide a better description for a shock wave as well as better mathematical properties, including existence, uniqueness, regularity, and ease of numerical scheme [12]. For consistency, a shock layer defined by the viscous NSF model, with vanishing viscosity and heat conductivity, should converge to the shock wave defined by the corresponding Euler equations in an appropriate sense. According to Dafermos [1, pp. 517–543], this is valid for the one-dimensional condition, where the solution of the equations with dissipation can be shown to converge boundedly almost everywhere to the solution of the inviscid equations. However, the situation becomes rather complicated for higher dimensionality, since the present mathematical theory for Euler systems in several space dimensions remains unresolved. Rigorous theory is only available for special cases. Majda [6, pp. 111–155] has investigated the existence and stability of shock solution of Euler systems, and Guès et al. [11] have investigated the existence and stability of a shock layer which converges to Majda's inviscid shock with vanishing viscosity.

This basic definition provides no practical criterion for a shock wave in a real flow field, where diffusion effects always exist,

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because it is cumbersome or impossible to verify the convergence condition for a specific flow field. Several methods for detection of shocks in a given flow field have been proposed, and Wu et al. [13] classified shock wave detection methods as four types. A straightforward notion is to determine shock wave by contours that the shock waves locate at the area with extremely dense pressure contours. This method is frequently used qualitatively by direct reading of contours, and can also be realized quantitatively by algorithms [14,15], which determine the shock wave by this sharply changing feature. More subtle methods based on the physical features of flow fields near shocks include density gradient [16], normal Mach number [17], and characteristics [18,19]. However, all these methods cannot be derived from the governing equations and basic definitions in a rigorous mathematical manner.

From both theoretical and practical aspects, a connection is required between the inviscid and viscous systems. Dynamical system theory incorporates the qualitative properties of differential equations and provides powerful tools for analysis of non-linear equations. There are numerous papers dealing with conservation laws employing the dynamical system approach. Zhang et al. investigate shock waves in Burgers' equation from approximate inertial manifolds [20] and bifurcation theory [21]. Kasimov [22] studies the stationary circular hydraulic jump via dynamical system theory. Hong et al. [23–25] discuss all steady states, including a stationary shock wave in a quasi-one-dimensional nozzle with geometric singular perturbation method based on the canard theory [26]. Inviscid and viscous systems were investigated, and the viscous system was treated as the singular perturbed system of the inviscid system. The small scale diffusion in the shock wave and the large scale quantities of the flow field were incorporated in a slow–fast system derived from the singular perturbed system. For flow fields with relatively small flow variable gradient, the behavior is dominated by the slow system, where viscous effects can be neglected; whereas for regions with extremely large gradient, the fast system, where viscous effects dominate, plays the central role. Thus, the shock wave is recognized as a fast motion, and a finer description of shock wave can be derived than using the inviscid Euler equations including the Rankine–Hugoniot condition. A similar but more complicated problem was studied by Fan and Lin [27,28], where the phase transition between liquid and vapor was included. They showed the existence and some properties of a fast evaporation layer. Both Hong et al. [23–25] and Fan and Lin treat the shock layer problem as a slow–fast system and resolve the two scale motion outside and inside the shock layer as the slow and fast systems. Thus, the shock wave in the viscous system can be defined as the fast motion in a slow–fast system. Nevertheless, they only discussed the one-dimensional case, or higher dimensionality with some special symmetry conditions.

In the present study, Hong et al.'s formulation [23–25] is extended to higher dimensionality. Compared with the one-dimensional case, the major difficulty for this extension is that the governing equations are no longer ordinary differential equations (ODEs), but more complicated partial differential equations (PDEs), and dynamical system theory is not as successful in the PDE arena. Indeed, existing theory on slow–fast systems concentrates on ODE systems and there is no similar theory on multidimensional gas dynamics. Accordingly, the present paper does not intend to investigate all the mathematical details of the existence, uniqueness, and stability of the solution. We outline the proof of existence of the solution using Lipschitz construction, similar to Lions' proof [29] of the existence and regularity of general compressible flow. The C^∞ smooth ODE system is constructed based on the mollifications of the Lipschitz continuous solution of PDEs. Properties of the ODE system are then studied using the geometric singular perturbation theory and an explicit criterion is proposed to identify the shock wave in the planar steady flow.

Section 2 discusses the governing equations and the important streamline flattening technique is introduced. Geometric singular perturbation theory is applied to the system in Section 3, and dynamical properties of fluid flow along each streamline are investigated. In Section 4, the theory developed in Section 3 is employed to study shock wave detecting techniques. Section 5 provides our final conclusions. Proofs and calculations are included in the appendices. Existence and regularity proved in Appendix A, the Lipschitz modification is introduced in Appendix B, and some calculations are completed in Appendix C.

2. Basic equations

2.1. Physical assumptions

A proper interpretation of diffusion effects, a modified NSF model, will provide a shock wave structure very close to experiments [3–5]. However, when dealing with macroscopic flow field, such a fine resolution of shock structure is unnecessary because varying the diffusion effects in an NSF model will only change the shock structure quantitatively. For example, Gilbarg [8] concluded that the NSF model omitting heat diffusion can provide qualitatively correct shock structure, which is adequate for the macroscopic flow study. The flow is usually assumed to be one-dimensional in shock structure studies where shear stresses play the same role as bulk stresses. When extending to the two-dimensional case, shear stresses may lead to complex vortex structures and bring extra difficulties. Indeed, existence and regularity theory and other mathematical problems of the Navier–Stokes equation are open questions, and somewhat more difficult to solve. On the other hand, the Euler equation without diffusion effects is difficult to solve due to poor regularity. Hence, an intermediate model is required, where some diffusion effects are included to smooth the solution, while other effects are omitted to exclude difficult problems.

The physical model for the present study was based on the discussion in Batchelor [30, pp. 151–154]. The main diffusive process in the shock wave is due to thermodynamical non-equilibrium, which is the departure of mechanical pressure, p, from the thermodynamic pressure or equilibrium pressure, p_e. In sudden expansion or compression of a fluid particle, mechanical pressure, which is defined as the negative mean normal stress, changes immediately. However, the thermodynamic state of the particle cannot change immediately and requires a relaxation process. This introduces an extra variable to the system and an extra equation governing the non-equilibrium relaxation process is required. For this, the pressure departure is assumed to be proportional to the velocity divergence,

$$p - p_e = -\mu_v \partial_{x_i} u_i, \tag{1}$$

where the summation convention is used; and μ_v, $i = 1, 2, 3$, x_i, u_i represent the bulk viscosity, the three directions, spatial coordinates, and the velocity components, respectively. Note that this non-equilibrium model is an approximation for the widely used treatment based on relaxation time, τ, in which the relaxation is added to the governing equation by a source term, such as $(p - p_e)/\tau$. An extra conservation law should be introduced for the non-equilibrium variable. It was shown in [31,32] that the source term model with an extra conservation law can be well approximated by a viscous term, such as (1). Thus, the relatively simple non-equilibrium model from Batchelor [30] was chosen for the present study.

The gas is assumed to be inviscid and obey the ideal gas law, $p_e = \rho RT$, where R, T are the positive gas constant and temperature, respectively, with heat capacity ratio $\gamma = 1.4$; and the mechanical pressure, p, is not equal to the thermodynamic pressure,