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Simple analytic approximations for the Blasius problem

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HIGHLIGHTS

- We derive analytic approximations for the Blasius function f and its derivatives.
- We extend the integral iteration scheme for the Blasius problem devised by H. Weyl.
- We compute very accurate bounds for the second derivative of f at the origin.
- We discuss the new approximations in the context of generalized Padè theory.

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ABSTRACT

The classical boundary layer problem formulated by Heinrich Blasius more than a century ago is revisited, with the purpose of deriving simple and accurate analytical approximations to its solution. This is achieved through the combined use of a generalized Padé approach and of an integral iteration scheme devised by Hermann Weyl. The iteration scheme is also used to derive very accurate bounds for the value of the second derivative of the Blasius function at the origin, which plays a crucial role in this problem.

1. Introduction

The Blasius problem is a prototypal example illustrating the development of a boundary layer as a result of the interaction between a weakly viscous flow and an obstacle, a phenomenon that is ubiquitous in nature. By now, it has more than a century of life, during which it has developed a vast bibliography; detailed information about this problem, together with a reminder of the essential literature, can be found in the recent review [1].

Here we only sketch the basic physical picture. Consider a steady, uniform flow impinging on a thin, wide plate parallel to it, and choose a (X, Y) Cartesian coordinate frame such that the flow points in the positive X direction and the plate is placed along the half-plane $Y = 0, X > 0$. For $X > 0$, the stream is basically undisturbed at large Y, but the flow must slow down when approaching the plate, because the fluid velocity must be zero at the solid boundary. In a low viscosity fluid, such as air or water, this only happens in a narrow region near the plate, named "boundary layer", or "shear layer", in which viscosity effects become essential. The main problem then is to compute the flow in this layer, and to match it with the flow outside.

Scale analysis of the steady-state Navier–Stokes equations shows that, in a first approximation, pressure gradients can be neglected within the boundary layer, so that the dominant balance, in the longitudinal (X) direction, is between the nonlinear advection terms and the leading viscosity term,

$$u\frac{\partial u}{\partial X} + v\frac{\partial u}{\partial Y} = \nu\frac{\partial^2 u}{\partial Y^2},$$

where (u, v) are the (X, Y) components of the flow, and ν is the kinematic viscosity. Together with mass continuity, $\partial_X u + \partial_Y v = 0$, this equation can be solved to yield the steady flow components in the boundary layer. Alternatively, the equilibrium equations can be reduced to a single third-order, nonlinear partial differential equation (PDE) for the flow streamfunction (see, e.g., [1]).

It turns out, however, that the problem can be further simplified, thanks to the existence of a continuous group of invariance,

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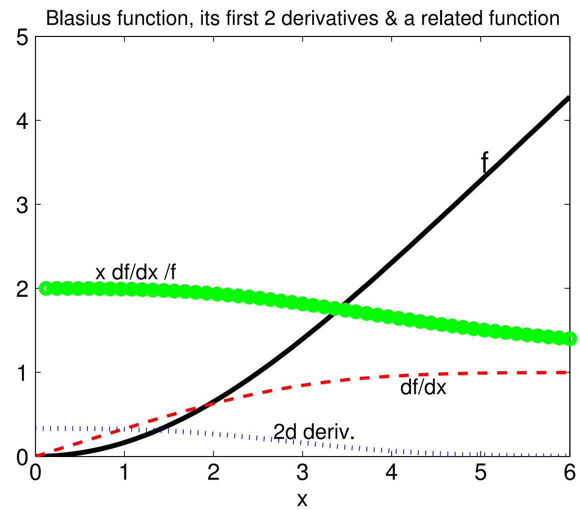

Fig. 1. The Blasius function $f(x)$ [solid black] is plotted with its first derivative [red dashed] and second derivative [blue dots]. The function $(x/f(x))df/dx$, which has a starring role in later derivations, is the thick green curve with markers.

pointed out by Blasius himself. This implies that the streamfunction is not a function of X and Y separately, but is instead a function of a similarity variable $x = Y/X^{1/2}$. As a consequence, the problem becomes one-dimensional, with the third-order PDE reducing to a third-order, nonlinear ordinary differential equation (ODE),

$$2f_{xxx} + ff_{xx} = 0, \tag{1}$$

which is referred to as the Blasius equation (BE hereafter). Here, f is related to the streamfunction of the two-dimensional problem, and the subscript denotes derivation with respect to the similarity variable x.

Eq. (1) needs to be solved subject to boundary conditions both at the origin and at infinity (see [1] for further details):

$$f(0) = f_x(0) = 0, \qquad f_x(\infty) = 1. \tag{2}$$

The boundary problem (1)–(2) was solved in [2] using the Chebyshev pseudospectral method. Fig. 1 shows the resulting profiles of $f(x)$ and of its first two derivatives, together with that of the function xf_x/f, which will be useful in the developments to follow. This very accurate numerical solution – errors for f and its first three derivatives are smaller than 10^{-9} over the whole real axis – will be used as a reference in the present work. All the functions plotted in Fig. 1 are smooth and monotonic. The Blasius function f has a simple structure: it grows like x^2 at small x, and asymptotes to x plus a negative constant at large x.

Despite the simple shape of the solution and the long history of the subject, analytic insight on the BE is still limited. While an accurate approximation for f_{xx} is known (see the next section), to the best of our knowledge, nobody has yet been able to provide a simple analytic approximation for $f(x)$ of comparable accuracy. Constructing such an approximation is the main objective of the present note.

2. Some basic results

The series expansion of f about the origin,

$$f(x) \approx \frac{\kappa}{2}x^2 - \frac{\kappa^2}{240}x^5 + \frac{11\kappa^3}{161280}x^8 - \frac{5\kappa^4}{4257792}x^{11} + \cdots, \tag{3}$$

was computed by Blasius himself, who derived a recurrence relation for the coefficients of the expansion. The coefficients depend on the parameter

$$\kappa \equiv f_{xx}(0), \tag{4}$$

which also has a physical meaning, being proportional to the shear stress on the plate. The parameter is *a priori* unknown, and not easy to compute analytically (we will come back to this issue later on). However, a second group invariance of the BE, discovered by Töpfer [3], allows for an efficient numerical computation of κ, which is approximately given by $\kappa \approx 0.332057336$. High precision values of κ and of the other constants that enter the Blasius problem can be found in Table 4.1 of [1].

Although the existence of a complex pole had long been known [4], a detailed understanding of the structure of the singularities of the BE in the complex plane has been achieved much more recently [2]. Among other results, it was found that the three singularities closest to the origin, which determine the radius of the convergence of the series expansion, lie on a circle of radius S, where $S \approx 5.69$. It was also shown in [2] that the series can be accelerated using Euler's summation, and that the accelerated series appears to converge everywhere on the positive x axis, which is the physical domain of the flow.

Treating the BE as a linear, first order ODE for f_{xx}, yields the well-known expression

$$f_{xx} = \kappa \exp\left(-\frac{1}{2}\int_0^x d\xi\, f(\xi)\right), \tag{5}$$

which explains the fast decay of f_{xx} with x shown by Fig. 1.

Expression (5) also suggests that a guess for f that is good at small x should give an approximation for f_{xx} accurate over a wider x-range, because the main contribution in the rhs of (5) comes from small x. One could place the series expansion (3) in (5), as Bairstow [5] did, but would then be confronted with the fact that the series has a finite radius of convergence. Bairstow noted, however, that taking $f \approx (\kappa/2)x^2$ yields

$$f_{xx}^{CG} \equiv f_{xx} = \kappa \exp\left(-\frac{\kappa}{12}x^3\right), \tag{6}$$

where CG stands for cubic-Gaussian. The approximation (6) is fairly accurate for x not too large: it is still within 1% from the numerical solution at $x = 2.5$, whereas the guess for f only preserves a similar accuracy up to $x = 1.5$.

It turns out that Bairstow's approach can be refined, to build a much better approximation for f_{xx},

$$f_{xx}^{PBS} \equiv f_{xx} = \kappa \frac{\exp(-0.0125151702320 88x^3)}{(1 + 0.005052091484171x^3)^3}, \tag{7}$$

which will be used here to test the accuracy of the approximations to be derived. The – non-trivial – derivation of this reference approximation, due to Parlange, Braddock and Sander [6], is sketched in the review [1].

Finally, we note that integration by parts of the identity $f(x) = \int_0^x d\xi f_\xi$ yields an integral expression for f in terms of its second derivative,

$$f(x) = \int_0^x d\xi\,(x - \xi) f_{\xi\xi}(\xi). \tag{8}$$

This suggests an iterative approach based on (5) and (8): given a guess for f that satisfies the boundary conditions at the origin, one could compute f_{xx} through (5), then get a new guess for f using (8), and so on. Would this approach prove convergent, it could be used to derive accurate numerical approximations to the solution of the BE.

3. Weyl's integral approach

The idea of iterating between f and f_{xx} is not new. It goes back to a trio of articles on the BE and related boundary layer problems that Hermann Weyl published three quarters of a century ago.