Physica D 310 (2015) 95–103

Contents lists available at ScienceDirect

Physica D

journal homepage: www.elsevier.com/locate/physd

Stability of heteroclinic cycles in transverse bifurcations

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HIGHLIGHTS

- We describe non-asymptotic stability of simple robust heteroclinic cycles.
- Essential asymptotic stability is equivalent to positive local stability indices.
- Almost complete instability is equivalent to negative local stability indices.
- Stability changes in transverse bifurcations differ for cycles of types A, B, C.

ARTICLE INFO

Article history:
Received 8 June 2015
Received in revised form
11 August 2015
Accepted 12 August 2015
Available online 20 August 2015
Communicated by I. Melbourne

Keywords:
Equivariant dynamics
Heteroclinic cycle
(Non-asymptotic) stability
Stability index

ABSTRACT

Heteroclinic cycles and networks exist robustly in dynamical systems with symmetry. They can be asymptotically stable, and gradually lose this stability through a variety of bifurcations, displaying different forms of non-asymptotic stability along the way. We analyse the stability change in a *transverse bifurcation* for different types of simple cycles in $\mathbb{R}^4$. This is done by first showing how stability of the cycle or network as a whole is related to stability indices along its connections — in particular, essential asymptotic stability is equivalent to all local stability indices being positive. Then we study the change of the stability indices. We find that all cycles of types B and C are generically essentially asymptotically stable after a transverse bifurcation, and that no type B cycle can be almost completely unstable (unlike type C cycles).

1. Introduction

In general dynamical systems heteroclinic cycles are not a generic phenomenon, being of possibly high codimension due to the nontransversal intersection of invariant manifolds. However, in systems with symmetry they can exist robustly. They are associated with intermittent dynamics, where long periods of seemingly stationary behaviour are interrupted by short intervals of drastic change. Asymptotic stability of a large class of cycles is well understood, see the work of Krupa and Melbourne [1–3]. Melbourne [4] was the first to discover that heteroclinic cycles often exhibit more diverse stability properties than the classic dichotomy between asymptotic stability and complete instability, which is typical, for instance, of hyperbolic equilibria. He defined *essential asymptotic stability (e.a.s.)* to describe attraction of a large measure set that is not a full neighbourhood. More recently, Podvigina and Ashwin [5] introduced stability indices σ and σ_{loc} as tools for quantifying stability and attraction along a trajectory.[1]

In this paper we study the different forms of non-asymptotic stability that heteroclinic cycles display as a transverse eigenvalue becomes positive. It is structured as follows. In Section 2 we recall the well-known setting in which heteroclinic cycles occur as robust phenomena and provide relevant definitions of stability properties and indices. Then, in Section 3 we show how essential asymptotic stability and its unstable counterpart *almost complete instability* are related to the index σ_{loc}, our main result being that for a heteroclinic cycle or network $X \subset \mathbb{R}^n$ the following holds (Theorem 3.1):

(i) X is e.a.s. $\Leftrightarrow$ $\sigma_{\text{loc}} > 0$ along all connecting trajectories.

(ii) X is a.c.u. $\Leftrightarrow$ $\sigma_{\text{loc}} < 0$ along all connecting trajectories.

In Section 4 we apply our results to heteroclinic cycles in $\mathbb{R}^4$ to obtain a complete picture of stability configurations during transverse bifurcations for the cycles classified as *simple* in [3]. This yields general results about non-asymptotic stability of simple cycles in $\mathbb{R}^4$.

2. Preliminaries

Consider a vector field on $\mathbb{R}^n$ given through a smooth differential equation $\dot{y} = f(y)$, where f is Γ-equivariant under the

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[1] Note that they use *predominant asymptotic stability (p.a.s.)* for the same attraction property that Melbourne [4] called e.a.s.

http://dx.doi.org/10.1016/j.physd.2015.08.005
0167-2789/

action of a finite group $\Gamma \subset O(n)$, that is,

$$f(\gamma.y) = \gamma.f(y), \quad \forall \gamma \in \Gamma \, \forall y \in \mathbb{R}^n.$$

A *heteroclinic cycle* is a collection of finitely many equilibria ξ_i, $i = 1, \ldots, m$, together with trajectories connecting them:

$$[\xi_i \to \xi_{i+1}] \subset W^u(\xi_i) \cap W^s(\xi_{i+1}) \neq \emptyset.$$

We set $\xi_{m+1} = \xi_1$ and write X to represent the heteroclinic cycle, i.e. the union of equilibria and connections. It is well-known that if the connections $[\xi_i \to \xi_{i+1}]$ are of saddle–sink type in a fixed-point subspace, then the cycle persists under perturbations respecting the Γ-equivariance and is called *robust*.

In the simplest case these fixed-point spaces are two-dimensional. We slightly adapt the definition of [3, p. 1181]: let $\Sigma_j \subset \Gamma$ be an isotropy subgroup and $P_j = \mathrm{Fix}(\Sigma_j)$. Assume that for all $j = 1, \ldots, m$ the connection $[\xi_j \to \xi_{j+1}]$ is a saddle–sink connection in P_j. Write $L_j = P_{j-1} \cap P_j$. A robust heteroclinic cycle $X \subset \mathbb{R}^4 \setminus \{0\}$ is called *simple* if

(i) $\dim(P_j) = 2$ for each j,
(ii) X intersects each connected component of $L_j \setminus \{0\}$ in at most one point,
(iii) the linearisation $df(\xi_j)$ has no double eigenvalues.

It is these cycles that we focus our attention on in Section 4. Note that condition (iii) was not part of the definition in [3], but seems to have been silently assumed in most of the literature. This was noticed by Podvigina and Chossat [6] who subsequently introduced the term *pseudo-simple* for cycles fulfilling only (i) and (ii).

Chossat et al. [7] classify simple cycles in $\mathbb{R}^4$ into types A, B and C and study bifurcations for each type. The same partitioning is also used in the context of asymptotic stability by Krupa and Melbourne [3] as well as Podvigina and Ashwin [5]. We reproduce this classification here from [3].

Definition 2.1 (*[3, Definition 3.2]*). Let $X \subset \mathbb{R}^4$ be a simple robust heteroclinic cycle.

(i) X is of *type A* if $\Sigma_j \cong \mathbb{Z}_2$ for all j.
(ii) X is of *type B* if there is a three-dimensional fixed-point subspace Q with $X \subset Q$.
(iii) X is of *type C* if it is not of type A or B.

All simple cycles of types B and C in $\mathbb{R}^4$ are enumerated in [3, Section 3(b)]. We recall their result in the next lemma, employing the usual notation $B_m^\pm$ and $C_m^\pm$, where m indicates the number of equilibria in the cycle and the superscript $\pm$ gives information on the symmetry group Γ, denoting whether $-\mathbb{1} \in \Gamma(-)$ or $-\mathbb{1} \notin \Gamma(+)$. For example, a B_3^- cycle has three equilibria and $-\mathbb{1} \in \Gamma$, while a B_2^+ cycle consists of two equilibria and $-\mathbb{1} \notin \Gamma$.

Lemma 2.2 (*[3]*). *There are seven distinct simple heteroclinic cycles of types B and C in $\mathbb{R}^4$ and the only finite groups $\Gamma \subset O(n)$ that allow them are the ones denoted in parentheses:*

- $B_1^+(\mathbb{Z}_2 \ltimes \mathbb{Z}_2^3)$, $B_2^+(\mathbb{Z}_2^3)$, $B_1^-(\mathbb{Z}_3 \ltimes \mathbb{Z}_2^4)$, $B_3^-(\mathbb{Z}_2^4)$
- $C_1^-(\mathbb{Z}_4 \ltimes \mathbb{Z}_2^4)$, $C_2^-(\mathbb{Z}_2 \ltimes \mathbb{Z}_2^4)$, $C_4^-(\mathbb{Z}_2^4)$.

Krupa and Melbourne [1,3] derive criteria for asymptotic stability of cycles in $\mathbb{R}^n$ (with a suitable generalisation of types A, B and C) depending on the eigenvalues of the vector field at each equilibrium. In a *heteroclinic network* (a connected union of more than one cycle), none of the individual cycles is asymptotically stable due to the presence of a connection with at least one other cycle. This gives rise to interesting dynamics regarding competition between cycles in a network as studied in [8] and creates a need for intermediate notions of stability—in particular that of *essential asymptotic stability* as introduced by Melbourne [4] and Brannath [9], where Brannath corrects a small inaccuracy in Melbourne's definition. We give a short overview of these and other stability concepts that we shall use.

In the rest of this work, let $B_\varepsilon(X)$ be an ε-neighbourhood of a set $X \subset \mathbb{R}^n$. We write $\mathcal{B}(X)$ for the basin of attraction of X, i.e. the set of points $x \in \mathbb{R}^n$ with $\omega(x) \subset X$. For $\delta > 0$ the δ-local basin of attraction is

$$\mathcal{B}_\delta(X) := \{x \in \mathcal{B}(X) \mid \forall t > 0\colon \phi_t(x) \in B_\delta(X)\},$$

where $\phi_t(.)$ is the flow generated by the system of equations. By $\ell(.)$ we denote Lebesgue measure, using a subscript to indicate the respective dimension where necessary.

As a counterpart to asymptotic stability (a.s.) we recall the notion of complete instability from [2].

Definition 2.3 (*[2, Definition 1.2]*). A compact invariant set X is called *completely unstable (c.u.)* if there is a neighbourhood U and a set D with $\ell(D) = 0$, such that for all $x \in U \setminus D$ there is $t_0 > 0$ with $\phi_{t_0}(x) \notin U$.

For the intermediate notions of stability we need the concept of relative (in)stability, extended from [10].

Definition 2.4. A compact invariant set X with $X \subset \overline{N}$ is called

1. *asymptotically stable relative to N* if for every neighbourhood U of X there is a neighbourhood V of X such that for all $x \in V \cap N$ we have $\omega(x) \subset X$ and $\phi_t(x) \in U$ for all $t > 0$.
2. *completely unstable relative to N* if there is a neighbourhood U of X such that for all $x \in U \cap N$ there is $t_0 > 0$ with $\phi_{t_0}(x) \notin U$.

Now essential asymptotic stability and almost complete instability can be formulated in terms of (in)stability relative to a large measure set.

Definition 2.5 (*[9, Definition 1.2]*). A compact invariant set X is called *essentially asymptotically stable (e.a.s.)* if it is asymptotically stable relative to a set $N \subset \mathbb{R}^n$ with the property that

$$\lim_{\varepsilon \to 0} \frac{\ell(B_\varepsilon(X) \cap N)}{\ell(B_\varepsilon(X))} = 1. \tag{1}$$

Definition 2.6 (*[2, Definition 1.2]*). A compact invariant set X is *almost completely unstable (a.c.u.)* if it is completely unstable relative to a set $N \subset \mathbb{R}^n$ with property (1).

Finally, we use *fragmentary asymptotic stability (f.a.s.)* from Podvigina [11] for any set with a positive measure basin of attraction.

In the next section, this terminology allows us to translate statements about stability of an entire cycle into statements about the stability indices along its connections and vice versa. We always state the strongest (in)stability property possible, i.e. when we say X is e.a.s. (a.c.u.) we implicitly mean that it is not a.s. (c.u.), and when we only call a set f.a.s. if it is neither e.a.s. nor a.c.u.

Podvigina and Ashwin [5] introduced the following stability index to quantify the attractiveness of a compact, invariant set X, Section 2.3 in [5].

Definition 2.7 (*[5, Definition 5]*). For $x \in X$ and $\varepsilon, \delta > 0$ define

$$\Sigma_\varepsilon(x) := \frac{\ell(B_\varepsilon(x) \cap \mathcal{B}(X))}{\ell(B_\varepsilon(x))}, \qquad \Sigma_{\varepsilon,\delta}(x) := \frac{\ell(B_\varepsilon(x) \cap \mathcal{B}_\delta(X))}{\ell(B_\varepsilon(x))}.$$

Then the *stability index* at x (with respect to X) is set to be

$$\sigma(x) := \sigma_+(x) - \sigma_-(x),$$