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Numerical bifurcation analysis of the bipedal spring-mass model

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HIGHLIGHTS

- The spring–mass model is transformed into a two-point boundary value problem (BVP).
- Calculation of stable solutions is reduced to the calculation of their boundaries.
- The complete solution manifold of the model is computed for the first time.
- Our approach can be extended for investigation of gait transitions and variability.
- Our approach is numerically stable. All BVPs can be solved using single shooting.

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ABSTRACT

The spring–mass model and its numerous extensions are currently one of the best candidates for templates of human and animal locomotion. However, with increasing complexity, their applications can become very time-consuming. In this paper, we present an approach that is based on the calculation of bifurcations in the bipedal spring–mass model for walking. Since the bifurcations limit the region of stable walking, locomotion can be studied by computing the corresponding boundaries. Originally, the model was implemented as a hybrid dynamical system. Our new approach consists of the transformation of the series of initial value problems on different intervals into a single boundary value problem. Using this technique, discontinuities can be avoided and sophisticated numerical methods for studying parametrized nonlinear boundary value problems can be applied. Thus, appropriate extended systems are used to compute transcritical and period-doubling bifurcation points as well as turning points. We show that the resulting boundary value problems can be solved by the simple shooting method with sufficient accuracy, making the application of the more extensive multiple shooting superfluous. The proposed approach is fast, robust to numerical perturbations and allows determining complete manifolds of periodic solutions of the original problem.

1. Introduction

The planar spring–mass model is an effective tool for investigating human and animal locomotion [1,2]. It was first developed for hopping and running [3,4] and was later extended to the bipedal model for walking [5]. Due to the complexity of the mechanics of human locomotion, its study often requires an extension of the model such as a segmentation of the leg spring [6,7], the upright trunk extension [8], the swing leg control [9] or the stance leg control [10]. The growing complexity of the model often results in considerably increased computational effort. Therefore, a new approach is required to work efficiently with the model in future.

Periodic solutions of the model mostly correspond to continuous locomotion patterns. In particular, stable periodic solutions are robust against small perturbations, which reduce the risk to fall [5]. Therefore, their study plays a very important role. In the bipedal spring–mass model, curves of periodic solutions are usually connected by bifurcations [11,12]. Moreover, these bifurcations also confine the regions of stable periodic walking [12]. Hence, it appears more appropriate and efficient to calculate a couple of bifurcations instead of computing the manifolds of periodic solutions.

Bifurcations are qualitative changes in the dynamics of the system, like vanishing stability or changing shape of the phase portrait. Bifurcation points are determined by parameter values, at which these changes happen [13,14]. They are also singularities in the mathematical model, which cannot be found by standard numerical techniques. Therefore, their computation requires a special approach, like the so called extended systems based on the well-known Lyapunov–Schmidt reduction [15–20]. Here, the original problem is embedded into a higher dimensional boundary

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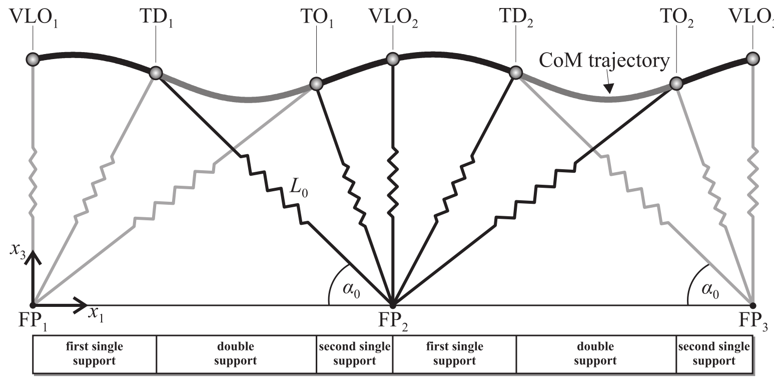

Fig. 1. Two subsequent steps of the bipedal spring–mass model for walking. The points on the centre of mass (CoM) trajectory show events of vertical leg orientation (VLO), touch-down (TD) and take-off (TO). Black and grey parts of the CoM trajectory represent single- and double-support phases, respectively. FP is the position of a foot point. The first step begins in VLO_1 at time t_0 and ends in VLO_2 at t_3. Events TD_1 and TO_1 occur at times t_1 and t_2. The second step ends in VLO_3.

value problem. Then, the solutions of this extended problem can be determined with numerical standard techniques like shooting methods.

Unlike the single-legged spring–mass model for running, the bipedal model provides all common kinds of bifurcations like turning points, secondary bifurcation points or Hopf bifurcations [11,12]. The bipedal spring–mass model has also a manifold of solutions [5,11]. However, only some of them are biologically relevant. A human walking gait is usually characterized by single- and double-support phases as well as by double-humped patterns of the vertical ground reaction force [21,22]. For this, we consider solutions of the bipedal spring–mass model with double-humped force patterns only, and define walking as the locomotion gait with at least one leg always having ground contact. Furthermore, we do not consider walking patterns with appearance of negative horizontal velocity, i.e. it is not allowed to walk backwards.

So far, the model was implemented as a hybrid dynamical system [23–25]. This implementation is described in Section 2. To apply appropriate techniques, we first transform the model into a two-point boundary value problem (BVP, Section 3). The resulting BVP and the corresponding extended systems for computation of bifurcations (Section 4) are solved using the well-approved software package RWPM [26–28].

2. The bipedal spring–mass model

2.1. Original implementation

The planar bipedal spring–mass model consists of two massless leg springs supporting the point mass m, which represents the centre of mass (CoM) of the human body [5]. Both leg springs have the same stiffness k_0 and rest length L_0. The location and velocity of the CoM in the real plane $\mathbb{R}^2$ are given by $(x_1, x_3)^T$ and $(x_2, x_4)^T$, respectively.

Any walking gait is completely characterized by four fundamental system parameters (the leg stiffness k_0, the angle of attack α_0, the leg length L_0, the system energy E_0) and the four-dimensional vector of initial conditions $(\hat{x}_1, \hat{x}_2, \hat{x}_3, \hat{x}_4)^T$ [5]. The calculation starts at the instant of vertical leg orientation (VLO, [11,29], Fig. 1), i.e. when the CoM is located directly above the foot point of the supporting leg during single support. The system is energy-conservative, i.e. the system energy E_0 remains constant during the whole step. In VLO, the system energy is given by

$$E_0 = mg\hat{x}_3 + \frac{m\left(\hat{x}_2^2 + \hat{x}_4^2\right)}{2} + \frac{k_0}{2}\left(L_0 - \hat{x}_3\right)^2. \tag{1}$$

Unless otherwise mentioned, we set $\hat{x}_1 := 0, L_0 := 1$ m, $m := 80$ kg and $k_0 = 16\,\text{kN}\,\text{m}^{-1}$. The system energy E_0 is varied to find families of periodic solutions.

One walking step comprises the first single-support phase, double-support phase and the second single-support phase (Fig. 1). The trajectory of the CoM in each phase is the solution of an initial value problem (IVP). Events of touch-down and take-off are transitions between the phases. The step begins in VLO_1 and ends in VLO_2.

In the following, we give an overview of this implementation of the model.

2.1.1. First single-support phase

The first single-support phase starts in the first instant of VLO. The IVP in this phase is given by

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= \frac{1}{m}k_0\left(L_0 - L_1(t)\right)\frac{x_1(t)}{L_1(t)} \\ \dot{x}_3(t) &= x_4(t) \\ \dot{x}_4(t) &= \frac{1}{m}\left(k_0\left(L_0 - L_1(t)\right)\frac{x_3(t)}{L_1(t)} - mg\right) \end{aligned} \tag{2}$$

and the vector of initial conditions is $x(t_0) = (\hat{x}_1, \hat{x}_2, \hat{x}_3, \hat{x}_4)^T$. Here, $L_1(t) := \sqrt{x_1^2(t) + x_3^2(t)}$ is the length of the first compressed leg spring during stance. The transition from the first single-support phase to the double-support phase (touch-down) happens at time t_1, when the landing condition $x_3(t_1) = L_0\sin(\alpha_0)$ is fulfilled (Fig. 1).

2.1.2. Double-support phase

The double-support phase starts at the time $t = t_1$. Here, the IVP is given by

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= \frac{1}{m}\left(k_0\left(L_0 - L_1(t)\right)\frac{x_1(t)}{L_1(t)} \right. \\ &\qquad \left. + k_0\left(L_0 - L_2(t)\right)\frac{x_1(t) - x_f}{L_2(t)}\right) \\ \dot{x}_3(t) &= x_4(t) \\ \dot{x}_4(t) &= \frac{1}{m}\left(k_0\left(L_0 - L_1(t)\right)\frac{x_3(t)}{L_1(t)} \right. \\ &\qquad \left. + k_0\left(L_0 - L_2(t)\right)\frac{x_3(t)}{L_2(t)} - mg\right), \end{aligned} \tag{3}$$

where $L_2(t) := \sqrt{(x_1(t) - x_f)^2 + x_3^2(t)}$ is the length of the second compressed leg spring and $(x_f, 0)$ is the position of the second foot