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Nonlinear mixed solitary—Shear waves and pulse equi-partition in a granular network

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HIGHLIGHTS

- We show a new type of mixed waves in coupled granular media.
- We show pulse equi-partition between two chains in theory and experiment.
- Analytical reduced models capture the main dynamics of exact numerical model.
- Nonlinear mapping technique is used to predict primary pulse transmission.

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ABSTRACT

We study primary pulse transmission in a two-dimensional granular network composed of two ordered chains that are nonlinearly coupled through Hertzian interactions. Impulsive excitation is applied to one of the chains (designated as 'excited chain'), and the resulting transmitted primary pulses in both chains are considered, especially in the non-directly excited chain (the 'absorbing chain'). A new type of mixed nonlinear solitary pulses–shear waves is predicted for this system, leading to primary pulse equi-partition between chains. An analytical reduced model for primary pulse transmission is derived to study the strongly nonlinear acoustics in the small-amplitude approximation. The model is re-scalable with energy and parameter-free, and is asymptotically solved by extending the one-dimensional nonlinear mapping technique of Starosvetsky (2012). The resulting nonlinear maps governing the amplitudes of the mixed-type waves accurately capture the primary pulse propagation in this system and predict the first occurrence of energy equipartition in the network. To confirm, in part, the theoretical results we experimentally test a series of two-dimensional granular networks, and prove the occurrence of strong energy exchanges leading to eventual pulse equi-partition between the excited and absorbing chains, provided that the number of beads is sufficiently large.

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1. Introduction

Ordered arrays of granular particles (beads) exhibit rich dynamical behavior, which has attracted considerable attention recently [1], due to their interesting properties, including passive adaptivity and tunability. This field of research originated from the original papers of Nesterenko [2] and Lazaridi, and Nesterenko [3], and further explored by other researchers [4–11].

It is well-known that the dynamics of uncompressed granular media is highly tunable, ranging from being strongly nonlinear and non-smooth in the absence of static pre-compression, to reducing to weakly nonlinear and smooth for large static pre-compression [7]. The nonlinearity in granular media arises from two sources: First, nonlinear Hertzian interactions between beads in contact, and second, bead separations in the absence of compressive forces between them leading to collisions between adjacent beads. Under certain simplifying assumptions, namely that the size of contact area is small and the developing strains are within the elastic limit, that the stress in the neighborhood of the contact is much higher than in the interiors of the beads, and that the time

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scale of dynamic interaction upon contact is much larger than the time of sound propagation in the bulk of the material of the beads, the nonlinear Hertzian dynamic interaction between beads can be modeled mathematically. This strongly nonlinear interaction between linearly elastic, spherical and non-dissipative beads may be expressed as,

$$F = k\, d_+^{3/2}$$

where d is the overlap displacement between beads, k is the coefficient of the Hertzian interaction, dependent on size and material, and F is the resulting force. The subscript $(+)$ indicates that the expression has meaning only if $d \geq 0$, and is equal to zero otherwise; in essence, (1) models both the nonlinear Hertzian interaction between beads in contact and the bead separation in the absence of compressive forces and loss of contact. The non-linearizable nature of (1) at the origin is a source of strong nonlinearity in the acoustics of uncompressed granular media [1]. Moreover, the fact that the force–displacement relation is zero for $d < 0$, enables separation between neighboring beads in the absence of compressive forces, and makes possible bead collisions which introduce nonsmoothness in the nonlinear acoustics. This represents an additional source of strong nonlinearity, and poses distinct challenges in the analytical treatment of the acoustics of ordered granular media.

When no applied pre-compression exists and separation between beads is possible, there is complete absence of linear acoustics in ordered granular media, which results in zero speed of sound as defined in the sense of linear acoustics through the classical wave equation; hence, the characterization of these media as 'sonic vacua' by Nesterenko [1]. Nevertheless, Nesterenko was the first to discover the propagation of a special class of solitary pulses in one-dimensional homogeneous granular chains that do not involve bead separations, and, hence, can be studied by asymptotic techniques in the continuum limit using long-wave approximations; in this work these solitary pulses will be denoted as *Nesterenko solitary pulses*. Furthermore, Nesterenko showed that these solitary pulses provide the fundamental mechanism for energy and momentum transfer in this class of ordered granular chains. With applied pre-compression a linear component in the acoustics is generated and the problem becomes linearizable, whereas for strong pre-compression and under the assumption of small amplitude oscillations, the dynamics of granular chains can be effectively described by the well-known FPU model [12–17] which in a long-wave approximation is reduced to the well-known integrable KdV model [1,9]. However, even in the strongly nonlinear regime (i.e., in the absence of any pre-compression) analytical techniques have been developed for studying solitary waves [3,5,18,19,9], traveling waves [20], traveling and standing breathers [21,22], targeted energy transfers [23], and other strongly nonlinear dynamical and acoustical phenomena. This indicates that this class of nonlinear sonic vacua can possess highly complex nonlinear dynamics, having no counterparts in linear theory.

The many current applications of ordered granular crystals include shock and vibration mitigation [24], shock energy trapping and absorption [25,8,26,27], acoustic focusing and lensing [10], tunability of solitary waves [7], frequency filtering [28], and various designs of new types of highly tunable and adaptive nonlinear acoustic metamaterials. It follows that understanding the nonlinear dynamical mechanisms of energy and momentum propagation, attenuation, and localization in multi-dimensional granular crystals is important for predictive design and utilization of this class of highly tunable media in engineering and applied physics applications.

In this work we numerically, analytically and experimentally examine primary wave transmission in a system of impulsively excited, coupled, finite granular chains, and show strong energy exchanges through excitation of transverse primary pulses (or shear waves) and the formation of a new type of mixed waves involving both solitary pulse propagation in the longitudinal direction and oscillatory shear waves in the orthogonal direction. We emphasize in this work that we are only interested in *primary pulse propagation*, that is, in the early time wave transmission of nonlinear waves, before any reflections from the axial boundaries of the system occur and secondary waves are generated. In that context, our analysis extends the one-dimensional nonlinear mapping technique developed by Starosvetsky [29], and Ben-Meir and Starosvetsky [30], and clarifies the strongly nonlinear dynamical mechanisms that govern the formation and evolution of these mixed primary waves. Moreover, our work explores a new way of frequency scattering of propagating solitary pulses in ordered granular media through the excitation of near-field shear waves, and contributes to the predictive design of multi-dimensional granular media in applications in practical acoustic metamaterial designs.

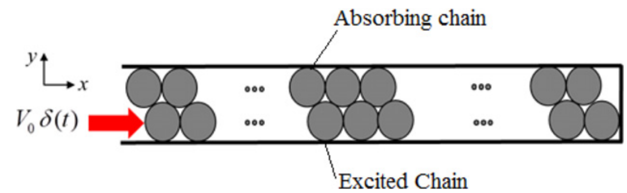

Fig. 1. Impulsively excited system of two coupled granular chains.

2. Computational study

The schematic diagram of the system of ordered granular chains considered is presented in Fig. 1. The system is composed of two coupled homogeneous granular chains with no prior compression. All beads are identical, spherical in shape, composed of linearly elastic material, and in point contact with each other. Strongly nonlinear Hertzian dynamic interactions between beads are assumed with no dissipative effects, such as those attributed to material damping, plasticity or dry friction. Hence, the considered system is Hamiltonian. One of the chains is excited by an impulse of intensity V_0 applied at $t = 0+$, and is designated as the 'excited chain', whereas the other is designated as the 'absorbing chain'. Rigid-wall boundary conditions are imposed on three sides of the system, with the fourth side being free, and the system is assumed to be at rest at $t = 0-$. As discussed below, these boundary conditions play an important role in the complex wave phenomena that evolve in this system.

Denoting by z_i and w_i the horizontal and vertical components, respectively, of the displacement of the ith bead of the excited chain, and by ξ_i and η_i the corresponding displacement components of the ith bead of the absorbing chain (with $i = 1, \ldots, n$), the strongly nonlinear and coupled equations of motion can be derived explicitly, and are listed in Appendix A (cf. Eqs. (A.1)). We note that this set of equations is not re-scalable with energy since not all terms are proportional to the same power of the displacement components. This means that the nonlinear dynamics of the system of Fig. 1 can change qualitatively with varying energy (i.e., with varying intensity of applied impulse). As discussed later, this can be rectified in the limit of small applied impulses, by deriving a reduced model which is fully re-scalable with energy (see Section 3). In any case, the intensity of the applied impulse should be sufficiently small in order that the resulting deformations of the elastic beads of the coupled granular chains conform with the assumptions necessary for the mathematical model (A.1), and the (small) rotations of the beads can be neglected.

In the equations of motion listed in Appendix A, $m = \frac{4}{3}\pi\rho R^3$ and R are the mass and radius of each bead, respectively; $\alpha = \frac{E\sqrt{2R}}{3(1-\nu^2)m}$ is the coefficient of the Hertzian interaction force, and λ