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Algebraic geometry and stability for integrable systems

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HIGHLIGHTS

- We present an algebro-geometric method of stability analysis for integrable systems.
- The method allows to conclude about stability from the shape of the spectral curve.
- We solve the stability problem for the free n-dimensional rigid body as an example.

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ABSTRACT

In 1970s, a method was developed for integration of nonlinear equations by means of algebraic geometry. Starting from a Lax representation with spectral parameter, the algebro-geometric method allows to solve the system explicitly in terms of theta functions of Riemann surfaces. However, the explicit formulas obtained in this way fail to answer qualitative questions such as whether a given singular solution is stable or not. In the present paper, the problem of stability for equilibrium points is considered, and it is shown that this problem can also be approached by means of algebraic geometry.

1. Introduction

As is well-known, many finite-dimensional integrable systems can be explicitly solved by means of algebraic geometry. The starting point for the algebro-geometric integration method is Lax representation. A dynamical system is said to admit a *Lax representation with spectral parameter* λ if the following two conditions are satisfied.

1. The phase space of the system can be identified with a certain submanifold $\mathcal{L}$ of the space $\mathfrak{gl}(n,\mathbb{C}) \otimes \mathbb{C}(\lambda)$ of matrix-valued functions of a complex variable λ.
2. Under this identification, equations of motion take the form

$$\frac{\mathrm{d}}{\mathrm{d}t}L_\lambda = [L_\lambda, A_\lambda(L_\lambda)] \quad (1)$$

where $L_\lambda \in \mathcal{L}$ is the phase variable, and A_λ is a mapping $A_\lambda\colon \mathcal{L} \to \mathfrak{gl}(n,\mathbb{C}) \otimes \mathbb{C}(\lambda)$.

Starting from a Lax representation with spectral parameter, the algebro-geometric integration method allows to write the solution of the system with initial data $L_\lambda = L_\lambda^0$ in terms of theta functions associated with the algebraic curve

$$\det(L_\lambda^0 - \mu \mathrm{E}) = 0, \quad (2)$$

which is called the *spectral curve*. We refer the reader to [1–5] and references therein for more details on the algebro-geometric integration method.

Despite the possibility to explicitly solve Eq. (1) in terms of theta functions, if we are interested in qualitative features of dynamics, theta-functional formulas seem to be of little use at least for the following reasons. Firstly, theta-functional solutions correspond to non-singular spectral curves, while most remarkable solutions, such as fixed points or stable periodic solutions, are related to degenerate curves. Secondly, theta-functional formulas provide solutions of the complexified system, and it is in general a difficult problem to describe real solutions. At the same time, many dynamical phenomena, such as stability, are related to the presence of a real structure.

In the present paper we study the Lyapunov stability problem for systems which admit a Lax representation with spectral parameter. We show that this problem can also be approached by means

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of classical algebraic geometry, and that this approach is very natural and fruitful. Though we focus on stability of equilibrium points, we expect that our results can be generalized to more general solutions. We note that the relation between topology of integrable systems and algebraic geometry has been extensively studied by M. Audin and her collaborators [5,6], however it seems that their approach cannot be directly used to study the stability problem.

Before we formulate the main result of the paper, let us describe the class of Lax matrices which we consider. Firstly, for the sake of simplicity, we restrict ourselves to the case when L_λ is polynomial in parameter λ, i.e. when $\mathcal{L} \subset \mathfrak{gl}(n, \mathbb{C}) \otimes \mathbb{C}[\lambda]$. Note that it is more standard to consider Lax matrices which are polynomial in λ and λ^{-1}, i.e. which belong to the loop algebra $\mathfrak{gl}(n, \mathbb{C}) \otimes \mathbb{C}[\lambda, \lambda^{-1}]$. However this situation can be reduced to the polynomial case by multiplying L_λ by a suitable power of λ.

Our second assumption is the following: there exists an anti-holomorphic involution $\tau: \mathbb{C} \to \mathbb{C}$ and a complex number σ of absolute value 1 such that for each $L_\lambda \in \mathcal{L}$

$$L_{\tau(\lambda)} = \sigma L^*_\lambda. \tag{3}$$

The presence of such involution is a common feature of many integrable systems. The most standard examples are provided in Table 1. See [7–11] for more details on these systems and their Lax representations.

Let us now formulate the main result. Let $L^0_\lambda \in \mathcal{L}$, and consider the associated spectral curve (2). The involution τ induces an anti-holomorphic involution

$$\widehat{\tau}: (\lambda, \mu) \mapsto (\tau(\lambda), \sigma\bar{\mu})$$

on the spectral curve. We show that if L^0_λ is a fixed point of the Lax equation (1), then, under some additional assumptions, a sufficient condition for its stability is that all singular points of the associated spectral curve are fixed points of $\widehat{\tau}$. If we interpret $\widehat{\tau}$ as a real structure, then this condition means that all singular points lie in the real part of the curve. We consider several examples in which this condition turns out to be necessary and sufficient.

Our first example is the Lagrange top. We use the algebro-geometric approach to recover the classical result that the rotation of a top is stable if the angular velocity is greater or equal than some critical value.

The second example is the top on a compact Lie algebras defined by a *sectional operator*. This system is related to the so-called *argument shift method*, see Mishchenko and Fomenko [12].

The third, and the most interesting, example is the free multidimensional rigid body, or Euler–Manakov top. It is a standard result that the rotation of a torque-free three-dimensional rigid body about the short or the long axis of inertia is stable, whereas the rotation about the middle axis is unstable. Using the algebro-geometric approach, we obtain a multidimensional generalization of this result. We note that this problem has previously been approached by different methods [13–18], however no complete solution has been known.

2. Stability for integrable and Lax systems

Let $\dot{x} = v(x)$ be a dynamical system on a manifold X, and assume that $f_1, \dots, f_N$ are its (in general, complex-valued) first integrals. The *moment map* is a map $F: X \to \mathbb{C}^N$ which maps $x \in X$ to $(f_1(x), \dots, f_N(x))$.

Proposition 2.1. *Assume that $x_0 \in X$ is an isolated point in the level set of the moment map. Then x_0 is Lyapunov stable fixed point of $\dot{x} = v(x)$.*

Table 1
Integrable systems which admit Lax representation with $L_{\tau(\lambda)} = \sigma L^*_\lambda$.

Integrable system	Lax matrix	τ	σ
Euler–Manakov top	$L_\lambda = A + B\lambda$, $A^* = -A, B^* = B$	$\lambda \mapsto -\bar{\lambda}$	-1
Kowalewski top	$L_\lambda = A + B\lambda + C\lambda^2$, $A^* = A, B^* = -B, C^* = C$	$\lambda \mapsto -\bar{\lambda}$	1
Geodesic flow on ellipsoid	$L_\lambda = A + B\lambda + C\lambda^2$, $A^* = A, B^* = -B, C^* = C$	$\lambda \mapsto -\bar{\lambda}$	1
Lagrange top	$L_\lambda = A + B\lambda + C\lambda^2$, $A^* = -A, B^* = -B, C^* = -C$	$\lambda \mapsto \bar{\lambda}$	-1

Proof. Let $x(t)$ be the solution with $x(0) = x_0$. Then $F(x(t)) = F(x(0))$, so $x(t) \in F^{-1}(F(x_0))$. Since x_0 is isolated in $F^{-1}(F(x_0))$, this implies that $x(t) = x_0$, i.e. x_0 is a fixed point. To prove stability, note that

$$f(x) = \sum_{i=1}^{N} |f_i(x) - f_i(x_0)|^2$$

is a Lyapunov function. $\square$

Remark 2.1. As was shown by Bolsinov and Borisov [19], a similar statement is true for periodic trajectories: if a periodic trajectory coincides with a connected component of the level set of the moment map, then it is stable. Moreover, under some additional assumptions, the converse is also true. In our case, the following is true. Let X be Poisson manifold, and let $\dot{x} = v(x)$ be a Hamiltonian system. Assume that $f_1, \dots, f_N$ is a complete family of analytic first integrals in involution. Further, assume that the level sets of F are compact, so that their connected components are invariant tori, and that $\dot{x} = v(x)$ is a non-resonant system, which means that its trajectories are dense on almost all tori, see Bolsinov and Fomenko [20]. Then the condition of Proposition 2.1 is necessary and sufficient.

Now, let us reformulate Proposition 2.1 for systems which admit a Lax representation with spectral parameter. Consider the space

$$\mathcal{P}_{m,n} = \{L_\lambda = B_m\lambda^m + \dots + B_0 \in \mathfrak{gl}(n, \mathbb{C}) \otimes \mathbb{C}[\lambda]\}$$

of $\mathfrak{gl}(n, \mathbb{C})$-valued polynomials of degree m, and let $\mathcal{L} \subset \mathcal{P}_{m,n}$ be its submanifold. Let A_λ be a map $A_\lambda: \mathcal{L} \to \mathfrak{gl}(n, \mathbb{C}) \otimes \mathbb{C}(\lambda)$, and assume that $\mathcal{L}$ is invariant with respect to the flow

$$\frac{\mathrm{d}}{\mathrm{d}t} L_\lambda = [L_\lambda, A_\lambda(L_\lambda)]. \tag{4}$$

To each $L_\lambda \in \mathcal{L}$ we can assign its spectral curve, i.e. an affine algebraic curve $C(L_\lambda)$ given by the equation $P(\lambda, \mu) = 0$ where

$$P(\lambda, \mu) = \det(L_\lambda - \mu \mathrm{E})$$

is the characteristic polynomial of L_λ. The following is well-known.

Proposition 2.2. *Let $L_\lambda(t)$ be a solution of (4). Then the curve $C(L_\lambda(t))$ does not depend on t.*

Proof. Eq. (4) implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}(L_\lambda)^k = [(L_\lambda)^k, A_\lambda]$$

for any positive integer k. Therefore, the function $\mathrm{Tr}\,(L_\lambda)^k$ is an integral of motion for any values of k and λ, and so are the coefficients of the characteristic polynomial $P(\lambda, \mu)$. $\square$

Proposition 2.3. *Let $L^0_\lambda \in \mathcal{L}$, and assume that L^0_λ is an isolated point in the isospectral variety*

$$S(L^0_\lambda) = \{L_\lambda \in \mathcal{L} \mid C(L_\lambda) = C(L^0_\lambda)\}.$$

Then L^0_λ is a stable fixed point of (4).